

LOCAL ENTROPY THEORY OF A RANDOM DYNAMICAL SYSTEM

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ABSTRACT. In this paper we introduce and discuss the notion of a continuous bundle random dynamical system associated to an infinite countable discrete amenable group action.

Given such a system, and a monotone sub-additive invariant family of random “continuous” functions, we introduce the concept of local fiber topological pressure and establish a variational principle for it, compared to measure-theoretic entropy. We also discuss it in some special cases.

We apply these results to both topological and measure-theoretic entropy tuples, obtain a variational relationship and give applications to general topological dynamical systems, recovering and extending many recent results in local entropy theory.

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1. INTRODUCTION

A Kolmogorov system or K -system is an important notion in measure-theoretic ergodic theory which is in some senses at the opposite extreme from a system of zero entropy [62]. Blanchard (1992) set out to find analogues of this notion in topological dynamics, and introduced the notions of uniformly positive entropy (u.p.e.) and completely positive entropy (c.p.e.) for continuous \mathbb{Z} -actions [3]. By localizing the concepts of u.p.e. and c.p.e., he defined the notion of entropy pairs, and showed that a u.p.e. system is disjoint from all minimal zero entropy systems [4]. He also obtained the maximal zero entropy factor for any continuous \mathbb{Z} -action [7]. Subsequently, a considerable literature has grown up on the local entropy theory of \mathbb{Z} -actions [3, 4, 5, 6, 7, 21, 28, 29, 31, 33, 34, 35, 36, 40, 63, 71] (and the references in them). For a nice survey of the area see [32].

Now, for each continuous \mathbb{Z} -action (X, T) , there exists a T -invariant Borel probability measure μ on X such that the classical ergodic theory of μ is linked with the study of the entropy theory of (X, T) . However, for a countable discrete group G , this is not necessarily the case: the free group on two generators, F_2 has actions with no invariant measures. It is well known that if G is an amenable group there exist invariant Borel probability measures on X . The class of amenable groups includes all finite groups, solvable groups and compact groups.

The development of the theory of actions of a general amenable group G lagged somewhat behind that of \mathbb{Z} actions. However, a turning point was the pioneering paper of Ornstein and Weiss [59] which laid the foundations of the theory of amenable group actions. Rudolph and Weiss [64] solved a longstanding problem, extending the theory of K -actions to actions of a countable discrete amenable group and showing that they must be mixing of all orders. Using this result, Dooley and Golodets [18] proved that every free ergodic action of a countable discrete amenable group with completely positive entropy has countable Lebesgue spectrum. Another longstanding open problem is the generalization of pointwise convergence results for \mathbb{Z} to general amenable group actions. In [50] Lindenstrauss gave an answer to this, proving the pointwise ergodic theorem for general locally compact amenable group actions along Følner sequences (with some conditions), and extending the Shannon-McMillan-Breiman Theorem to all countable discrete amenable group actions.

Local entropy theory for infinite countable discrete amenable group actions has been systematically studied by Huang, Ye and Zhang [37]. Kerr and Li [40] studied independence of such actions using combinatorial methods. Global entropy theory for amenable group actions has also been discussed in [56]. For related work, see [13, 16, 17, 20, 25, 30, 42, 57, 58, 60, 65, 68, 69] (and the references therein) and Benjy Weiss' lovely survey article [70].

Our aim in this article is to extend the theory of local entropy to the setting of random dynamical systems. In this setting, rather than considering iterations of just one map, we study the successive application of different transformations chosen at random. The basic framework was established by Ulam and von Neumann [66] and later Kakutani [39] in proofs of the random ergodic theorem. During the 1980s, interest in the ergodic theory of random transformations grew, as the connection was made with stochastic flows which arise as solutions of stochastic differential equations. This area was first studied in the framework of the relativized ergodic theory of Ledrappier and Walters [48] and later in the theory of random

transformations, see [2, 8, 9, 10, 15, 41, 43, 44, 45, 46, 49, 52, 53, 54]. In particular, it was shown in [8] that the Abramov-Rokhlin mixed entropy of the fiber of a skew-product transformation ([1]) is the cornerstone for the theory of entropy of random transformations. Moreover, [8, 44] introduced the concept of topological pressure in the framework of a continuous bundle random dynamical system, as a real-valued map on the space of random “continuous” functions and a variational principle was established connecting it with measure-theoretic entropy. See also [75] for some related topics.

To date, most discussions of random dynamical systems concern \mathbb{R} -actions, \mathbb{Z} -actions or even \mathbb{Z}_+ -actions. Furthermore, to the best of our knowledge, there is little discussion of the local theory. Broadly speaking, our aim in this paper is to make a systematic study of the local entropy theory of a continuous bundle random dynamical system over an infinite countable discrete amenable group.

We shall extend the notion of a continuous bundle random dynamical system to the setting of an infinite countable discrete amenable group action and a monotone sub-additive invariant family of random “continuous” functions. We define the local fiber topological pressure for a finite measurable cover, and establish its basic properties. A key point in the local entropy theory of \mathbb{Z} -actions (and its general case [37]) is the local variational principle concerning topological and measure-theoretic entropy for finite open covers. In the case of a finite random “open” cover we establish a variational principle for local fiber topological pressure and measure-theoretic entropy. We discuss a special case, which shows that these assumptions are very natural. In particular, as corollaries of our local variational principle, we are able to obtain the main results in [8, 44, 53, 75]. We introduce and discuss both topological and measure-theoretical entropy tuples for a continuous bundle random dynamical system, and our local variational principle allows us to build a variational relationship between these two kinds of entropy tuples. Finally, we apply these results to the setting of a general topological dynamical system, extending many recent results in the local entropy theory of \mathbb{Z} -actions ([32]) and of infinite countable discrete amenable group actions ([37]) to the setting of random dynamical systems, and obtaining some new results even in the deterministic setting. There remain some unsolved questions, which stand as challenges to the further study of the topic.

Some ideas of the paper have been used in [19] to obtain sub-additive ergodic theorems for countable amenable groups.

The paper consists of three parts and is organized as follows.

The first part gives some preliminaries, on infinite countable discrete amenable groups following [59, 68, 70], on general measurable dynamical systems, and on continuous bundle random dynamical systems of an infinite countable discrete amenable group action extending the case for \mathbb{Z} [44, 45, 53]. In addition to recalling known results, this part contains new results: convergence results for infinite countable discrete amenable groups (Proposition 2.3 and Proposition 2.8, extending results from [56]), where the difference from the special case of \mathbb{Z} is seen in Example 2.9; the relative Pinsker formula for a measurable dynamical system of an infinite countable discrete amenable group action (Theorem 3.5 and Remark 3.6), discussed in [30] in the case where the state space is a Lebesgue space; further understanding the (local) entropy theory of general measurable dynamical systems (Theorem 3.13 and Question 3.14).

In the second part we present and prove our main results. More precisely, given a continuous bundle random dynamical system of an infinite countable discrete amenable group action and a monotone sub-additive invariant family of random “continuous” functions, in §5 following the ideas of [38, 63, 74] we introduce the local fiber topological pressure for a finite measurable cover, and discuss its basic properties; in §6 we introduce the concept of factor excellent and good covers, which are necessary assumptions underlying our main result, Theorem 7.1. We show in Theorem 6.9 and Theorem 6.10 that many interesting covers are included in this special class of finite measurable covers. In §7, we state Theorem 7.1 and give some remarks and direct applications, obtaining as corollaries, the main results in [8, 44, 53, 75]. In §8 we present the details of the proof of Theorem 7.1 following the ideas from [35, 37, 55, 74] and in §9 we discuss other assumptions appearing in Theorem 7.1.

In §10 we strengthen Theorem 7.1 in the special case of the infinite countable discrete amenable group admitting a tiling Følner sequence and obtain Theorem 10.2 and Corollary 10.3. We also discuss abelian group actions, showing that the assumptions for Theorem 7.1 are natural. Observe that for a continuous bundle random dynamical system over a \mathbb{Z} -action, and a real-valued random “continuous” function, Kifer ([44]) introduced the global fiber topological pressure using separated subsets with a positive constant and showed that the resulting pressure is the same if we use separated subsets with a positive random variable from a natural class. In §11 we give a general version of Theorem 7.1, which may be viewed as a (local) counterpart of Kifer’s result in our setting.

The third and last part of the paper is devoted to some applications of the local variational principle. In §12, following the ideas of [4, 6, 32, 34, 36, 37] (and the references therein), we introduce both topological and measure-theoretic entropy tuples for a continuous bundle random dynamical system in our setting, and build a variational relationship between them. Finally, in §13 we apply these results to the setting of a general topological dynamical system, incorporating and extending many recent results in the theory of local entropy for \mathbb{Z} -actions [4, 6, 32, 34, 36] and for an infinite countable discrete amenable group action from [37], as well as establishing some new results.

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Part 1. Preliminaries

Denote by $\mathbb{Z}, \mathbb{Z}_+, \mathbb{N}, \mathbb{R}, \mathbb{R}_+$ the set of all integers, non-negative integers, positive integers, real numbers, non-negative real numbers, respectively.

2. INFINITE COUNTABLE DISCRETE AMENABLE GROUPS

In this section, we recall the principal results from [56, 59, 68, 70] and obtain new convergence results for an infinite countable discrete amenable groups (Proposition 2.3 and Proposition 2.8). The difference of Proposition 2.3 and Proposition 2.8 is shown by Example 2.9 even in the setting of an infinite countable discrete amenable group admitting a tiling Følner sequence.

Let G be an infinite countable discrete group and denote by e_G the identity of G . Denote by \mathcal{F}_G the set of all non-empty finite subsets of G .

G is called *amenable*, if for each $K \in \mathcal{F}_G$ and any $\delta > 0$ there exists $F \in \mathcal{F}_G$ such that

$$|F \Delta KF| < \delta |F|,$$

where $|\bullet|$ is the counting measure of the set \bullet , $KF = \{kf : k \in K, f \in F\}$ and $F \Delta KF = (F \setminus KF) \cup (KF \setminus F)$. Let $K \in \mathcal{F}_G$ and $\delta > 0$. Set $K^{-1} = \{k^{-1} : k \in K\}$. $A \in \mathcal{F}_G$ is called (K, δ) -invariant, if

$$|K^{-1}A \cap K^{-1}(G \setminus A)| < \delta |A|.$$

A sequence $\{F_n : n \in \mathbb{N}\}$ in \mathcal{F}_G is called a *Følner sequence*, if for any $K \in \mathcal{F}_G$ and for any $\delta > 0$, F_n is (K, δ) -invariant whenever $n \in \mathbb{N}$ is sufficiently large, i.e., for each $g \in G$,

$$(2.1) \quad \lim_{n \rightarrow \infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0.$$

It is not hard to obtain from this the usual asymptotic invariance property: G is amenable if and only if G has a Følner sequence $\{F_n\}_{n \in \mathbb{N}}$.

For example, $G = \mathbb{Z}$ a Følner sequence is defined by $F_n = \{0, 1, \dots, n-1\}$, or, indeed, $\{a_n, a_n + 1, \dots, a_n + n - 1\}$ for any sequence $\{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{Z}$.

Throughout the current paper, we will assume that G is an infinite countable discrete amenable group.

The following terminology and results are due to Ornstein and Weiss [59] (see also [64, 68]).

Let $A_1, \dots, A_k, A \in \mathcal{F}_G$ and $\epsilon \in (0, 1)$, $\alpha \in (0, 1]$.

(1) Subsets A_1, \dots, A_k are ϵ -disjoint if there are $B_1, \dots, B_k \in \mathcal{F}_G$ such that

$$B_i \subseteq A_i, \frac{|B_i|}{|A_i|} > 1 - \epsilon \text{ and } B_i \cap B_j = \emptyset \text{ whenever } 1 \leq i \neq j \leq k.$$

(2) $\{A_1, \dots, A_k\}$ α -covers A if

$$\frac{|A \cap \bigcup_{i=1}^k A_i|}{|A|} \geq \alpha.$$

(3) A_1, \dots, A_k ϵ -quasi-tile A if there exist $C_1, \dots, C_k \in \mathcal{F}_G$ such that

- (a) for $i = 1, \dots, k$, $A_i C_i \subseteq A$ and $\{A_i c : c \in C_i\}$ forms an ϵ -disjoint family,
- (b) $A_i C_i \cap A_j C_j = \emptyset$ if $1 \leq i \neq j \leq k$ and

(c) $\{A_i C_i : i = 1, \dots, k\}$ forms a $(1 - \epsilon)$ -cover of A .

The subsets C_1, \dots, C_k are called the *tiling centers*.

We have (see for example [37, Proposition 2.3], [59] or [68, Theorem 2.6]).

Proposition 2.1. *Let $\{F_n : n \in \mathbb{N}\}$ and $\{F'_n : n \in \mathbb{N}\}$ be two Følner sequences of G . Assume that $e_G \in F_1 \subseteq F_2 \subseteq \dots$. Then for any $\epsilon \in (0, \frac{1}{4})$ and each $N \in \mathbb{N}$, there exist integers n_1, \dots, n_k with $N \leq n_1 < \dots < n_k$ such that F_{n_1}, \dots, F_{n_k} ϵ -quasi-tiling F'_m whenever m is large enough.*

Let $f : \mathcal{F}_G \rightarrow \mathbb{R}$ be a function. Following [37], we say that f is:

- (1) *monotone*, if $f(E) \leq f(F)$ for any $E, F \in \mathcal{F}_G$ satisfying $E \subseteq F$;
- (2) *non-negative*, if $f(F) \geq 0$ for any $F \in \mathcal{F}_G$;
- (3) *G -invariant*, if $f(Fg) = f(F)$ for any $F \in \mathcal{F}_G$ and $g \in G$;
- (4) *sub-additive*, if $f(E \cup F) \leq f(E) + f(F)$ for any $E, F \in \mathcal{F}_G$.

The following convergence property is well known (see for example [37, Lemma 2.4] or [51, Theorem 6.1]).

Proposition 2.2. *Let $f : \mathcal{F}_G \rightarrow \mathbb{R}$ be a monotone non-negative G -invariant sub-additive function. Then for any Følner sequence $\{F_n : n \in \mathbb{N}\}$ of G , the sequence $\{\frac{f(F_n)}{|F_n|} : n \in \mathbb{N}\}$ converges and the value of the limit is independent of the selection of the Følner sequence $\{F_n : n \in \mathbb{N}\}$.*

In fact, this result can be strengthened along two different lines as follows.

The first and stronger version of it is:

Proposition 2.3. *Let $f : \mathcal{F}_G \rightarrow \mathbb{R}$ be a function. Assume that $f(Eg) = f(E)$ and $f(E \cap F) + f(E \cup F) \leq f(E) + f(F)$ whenever $g \in G$ and $E, F \in \mathcal{F}_G$ (here, we set $f(\emptyset) = 0$ by convention). Then for any Følner sequence $\{F_n : n \in \mathbb{N}\}$ of G , we have that the sequence $\{\frac{f(F_n)}{|F_n|} : n \in \mathbb{N}\}$ converges and the value of the limit is independent of the selection of the Følner sequence $\{F_n : n \in \mathbb{N}\}$, in fact,*

$$\lim_{n \rightarrow \infty} \frac{f(F_n)}{|F_n|} = \inf_{F \in \mathcal{F}_G} \frac{f(F)}{|F|} \quad (\text{and so } = \inf_{n \in \mathbb{N}} \frac{f(F_n)}{|F_n|}).$$

Remark 2.4. *A version of this Proposition was proved by Moulin Ollagnier [56, Lemma 2.2.16 and Proposition 3.1.9]. However, observe that the definition of sub-additivity in [56, Definition 3.1.5] is slightly different from ours.*

We are grateful to Hanfeng Li and Benjy Weiss for pointing this out to us.

While our proof follows similar lines to that of [56], the details are somewhat different. We present a proof here, both for completeness and because we will need some of the ideas in Proposition 9.2 below.

In order to prove Proposition 2.3, we need the following two lemmas.

Lemma 2.5. *Let $T, E \in \mathcal{F}_G$. Then $\sum_{t \in T} 1_{tE} = \sum_{g \in E} 1_{Tg}$.*

Proof. Set $L = \sum_{t \in T} 1_{tE}$ and $R = \sum_{g \in E} 1_{Tg}$. Let $g' \in G$. Then $L(g') > 0$ if and only if there exists $t \in T$ such that $g' \in tE$, if and only if there exists $g \in E$ such that $g' \in Tg$, if and only if $R(g') > 0$. Moreover, for any given $n \in \mathbb{N}$, $L(g') = n$ if and only if there exist exactly n distinct elements t_1, \dots, t_n of T such that $g' \in t_i E$ (say $g' = t_i g_i$ for some $g_i \in E$) for each $i = 1, \dots, n$, if and only if there exist exactly n distinct elements g_1, \dots, g_n of E such that $g' \in Tg_i$ for each $i = 1, \dots, n$, if and only if $R(g') = n$. This finishes the proof. \square

We also need the following result. As Lemma 9.3 below is a general version of this Lemma, we shall defer its proof: see also [56, Lemma 2.2.16].

Lemma 2.6. *Let $f : \mathcal{F}_G \rightarrow \mathbb{R}$ be a function. Assume that $f(E \cap F) + f(E \cup F) \leq f(E) + f(F)$ whenever $E, F \in \mathcal{F}_G$ (here, we set $f(\emptyset) = 0$ by convention). If $E, E_1, \dots, E_n \in \mathcal{F}_G, n \in \mathbb{N}$ satisfy*

$$1_E = \sum_{i=1}^n a_i 1_{E_i},$$

where all $a_1, \dots, a_n > 0$ are rational numbers, then

$$f(E) \leq \sum_{i=1}^n a_i f(E_i).$$

Now we prove Proposition 2.3.

Proof of Proposition 2.3. Let $\{F_n : n \in \mathbb{N}\}$ be a Følner sequence for G . Observe that there exists $M \in \mathbb{R}$ such that $f(\{g\}) = M$ for each $g \in G$. Set

$$f' : \mathcal{F}_G \rightarrow \mathbb{R}, E \mapsto f(E) - |E|M \leq 0$$

for each $E \in \mathcal{F}_G$. The function $f' : \mathcal{F}_G \rightarrow \mathbb{R}$ satisfies $f'(Eg) = f'(E)$ and $f'(E \cap F) + f'(E \cup F) \leq f'(E) + f'(F)$ whenever $g \in G$ and $E, F \in \mathcal{F}_G$ (again, we set $f'(\emptyset) = 0$ by convention). Thus, we only need show that the sequence $\{\frac{f'(F_n)}{|F_n|} : n \in \mathbb{N}\}$ converges and

$$(2.2) \quad \lim_{n \rightarrow \infty} \frac{f'(F_n)}{|F_n|} = \inf_{F \in \mathcal{F}_G} \frac{f'(F)}{|F|}.$$

Obviously,

$$(2.3) \quad \liminf_{n \rightarrow \infty} \frac{f'(F_n)}{|F_n|} \geq \inf_{F \in \mathcal{F}_G} \frac{f'(F)}{|F|}.$$

For the other direction, let $T \in \mathcal{F}_G$ be fixed. As $\{F_n : n \in \mathbb{N}\}$ is a Følner sequence of G , for each $n \in \mathbb{N}$ we set $E_n = F_n \cap \bigcap_{g \in T} g^{-1}F_n \subseteq F_n$, then $\lim_{n \rightarrow \infty} \frac{|E_n|}{|F_n|} = 1$. Now using Lemma 2.5 one has that, for each $n \in \mathbb{N}$,

$$\sum_{t \in T} 1_{tE_n} = \sum_{g \in E_n} 1_{Tg}.$$

By the construction of E_n , $tE_n \subseteq F_n$ for any $t \in T$. Thus there exist $E'_1, \dots, E'_m \in \mathcal{F}_G, m \in \{0\} \cup \mathbb{N}$ and rational numbers $a_1, \dots, a_m > 0$ such that

$$1_{F_n} = \frac{1}{|T|} \sum_{t \in T} 1_{tE_n} + \sum_{j=1}^m a_j 1_{E'_j}.$$

Hence

$$(2.4) \quad 1_{F_n} = \frac{1}{|T|} \sum_{g \in E_n} 1_{Tg} + \sum_{j=1}^m a_j 1_{E'_j},$$

which implies that

$$\begin{aligned}
 f'(F_n) &\leq \frac{1}{|T|} \sum_{g \in E_n} f'(Tg) + \sum_{j=1}^m a_j f'(E'_j) \text{ (applying Lemma 2.6 to } f') \\
 (2.5) \quad &\leq \frac{1}{|T|} \sum_{g \in E_n} f'(T) = \frac{|E_n|}{|T|} f'(T) \\
 &\quad \text{(as the function } f' \text{ is } G\text{-invariant and negative).}
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \limsup_{n \rightarrow \infty} \frac{f'(F_n)}{|F_n|} &\leq \limsup_{n \rightarrow \infty} \frac{|E_n|}{|T|} \cdot \frac{f'(T)}{|F_n|} \text{ (using (2.5))} \\
 &= \frac{f'(T)}{|T|} \text{ (as } \lim_{n \rightarrow \infty} \frac{|E_n|}{|F_n|} = 1),
 \end{aligned}$$

which implies

$$(2.6) \quad \limsup_{n \rightarrow \infty} \frac{f'(F_n)}{|F_n|} \leq \inf_{F \in \mathcal{F}_G} \frac{f'(F)}{|F|}.$$

Now (2.2) follows directly from (2.3) and (2.6). This completes the proof. \square

Now we introduce a second stronger version of Proposition 2.2.

Let $\emptyset \neq T \subseteq G$. We say that T *tiles* G if there exists $\emptyset \neq G_T \subseteq G$ such that $\{Tc : c \in G_T\}$ forms a partition of G , that is, $Tc_1 \cap Tc_2 = \emptyset$ if c_1 and c_2 are different elements from G_T and $\bigcup_{c \in G_T} Tc = G$.

Denote by \mathcal{T}_G the set of all non-empty finite subsets of G which tile G . Observe that $\mathcal{T}_G \neq \emptyset$, as $\mathcal{T}_G \supseteq \{\{g\} : g \in G\}$.

By a *measurable dynamical G -system* (MDS) (Y, \mathcal{D}, ν, G) we mean a probability space (Y, \mathcal{D}, ν) and a group G of invertible measure-preserving transformations of (Y, \mathcal{D}, ν) with e_G acting as the identity transformation.

Let (Y, \mathcal{D}, ν, G) be an MDS. We say that G *acts freely* on (Y, \mathcal{D}, ν) if $\{y \in Y : gy = y\}$ has zero ν -measure for any $g \in G \setminus \{e_G\}$.

As shown by the following result, tiling sets play a key role in establishing a counterpart of Rokhlin's Lemma for infinite countable discrete amenable group actions (cf [70, Theorem 3.3 and Proposition 3.6]).

Proposition 2.7. *Let $T \in \mathcal{F}_G$. Then $T \in \mathcal{T}_G$ if and only if, for every MDS (Y, \mathcal{D}, ν, G) , where G acts freely on (Y, \mathcal{D}, ν) , for each $\epsilon > 0$ there exists $B \in \mathcal{D}$ such that the family $\{tB : t \in T\}$ are disjoint and $\nu(\bigcup_{t \in T} tB) \geq 1 - \epsilon$.*

The class of countable amenable groups admitting a *tiling Følner sequence* (i.e. a Følner sequence consisting of tiling subsets of the group) is large, and includes all countable amenable linear groups and all countable residually finite amenable groups [69]. Recall that a *linear group* is an abstract group which is isomorphic to a matrix group over a field K (i.e. a group consisting of invertible matrices over some field K); a group is *residually finite* if the intersection of all its normal subgroups of finite index is trivial. Note that any finitely generated nilpotent group is residually finite.

If the group admits a tiling Følner sequence, we have a stronger version of Proposition 2.2 (this strengthens [70, Theorem 5.9]).

Proposition 2.8. *Let $f : \mathcal{F}_G \rightarrow \mathbb{R}$ be a function. Assume that $f(Eg) = f(E)$ and $f(E \cup F) \leq f(E) + f(F)$ whenever $g \in G$ and $E, F \in \mathcal{F}_G$ satisfy $E \cap F = \emptyset$. Then for any tiling Følner sequence $\{F_n : n \in \mathbb{N}\}$ of G , the sequence $\{\frac{f(F_n)}{|F_n|} : n \in \mathbb{N}\}$ converges and the limit is independent of the selection of the tiling Følner sequence $\{F_n : n \in \mathbb{N}\}$, in fact:*

$$\lim_{n \rightarrow \infty} \frac{f(F_n)}{|F_n|} = \inf_{F \in \mathcal{T}_G} \frac{f(F)}{|F|} \text{ (and so } = \inf_{n \in \mathbb{N}} \frac{f(F_n)}{|F_n|}).$$

Proof. Let $\{F_n : n \in \mathbb{N}\}$ be a tiling Følner sequence for G . Then there exists $M \in \mathbb{R}$ such that $f(\{g\}) = M$ for each $g \in G$. Set

$$h : \mathcal{F}_G \rightarrow \mathbb{R}, E \mapsto |E|M - f(E)$$

for each $E \in \mathcal{F}_G$. The function $h : \mathcal{F}_G \rightarrow \mathbb{R}_+$ satisfies $h(Eg) = h(E)$ and $h(E \cup F) \geq h(E) + h(F)$ whenever $g \in G$ and $E, F \in \mathcal{F}_G$ satisfy $E \cap F = \emptyset$. Thus, we only need show that the sequence $\{\frac{h(F_n)}{|F_n|} : n \in \mathbb{N}\}$ converges and

$$(2.7) \quad \lim_{n \rightarrow \infty} \frac{h(F_n)}{|F_n|} = \sup_{F \in \mathcal{T}_G} \frac{h(F)}{|F|}.$$

It is clear that

$$(2.8) \quad \limsup_{n \rightarrow \infty} \frac{h(F_n)}{|F_n|} \leq \sup_{F \in \mathcal{T}_G} \frac{h(F)}{|F|}.$$

For the other direction, first let $\epsilon > 0$ and $F \in \mathcal{T}_G$ be fixed: G_F is a subset of G such that $\{Fg : g \in G_F\}$ forms a partition of G . As $\{F_n : n \in \mathbb{N}\}$ is a tiling Følner sequence of G , F_n is (F, ϵ) -invariant whenever $n \in \mathbb{N}$ is large enough. Now for each $n \in \mathbb{N}$ set $E'_n = \{g \in G_F : Fg \subseteq F_n\}$ and $E_n = \{g \in G_F : Fg \cap F_n \neq \emptyset\}$, one has

$$E_n \setminus E'_n \subseteq F^{-1}F_n \cap F^{-1}(G \setminus F_n).$$

Thus if $n \in \mathbb{N}$ is sufficiently large,

$$\frac{|F_n|}{|F|} \leq |E_n| \leq |E'_n| + \epsilon|F_n|, \text{ i.e. } |E'_n| \geq (\frac{1}{|F|} - \epsilon)|F_n|,$$

and thus

$$\frac{h(F_n)}{|F_n|} \geq \frac{h(FE'_n)}{|F_n|} \geq \frac{h(F)|E'_n|}{|F_n|} \geq (\frac{1}{|F|} - \epsilon)h(F).$$

This implies

$$\liminf_{n \rightarrow \infty} \frac{h(F_n)}{|F_n|} \geq (\frac{1}{|F|} - \epsilon)h(F).$$

Since both $\epsilon > 0$ and $F \in \mathcal{T}_G$ are arbitrary, one may conclude

$$(2.9) \quad \liminf_{n \rightarrow \infty} \frac{h(F_n)}{|F_n|} \geq \sup_{F \in \mathcal{T}_G} \frac{h(F)}{|F|}.$$

Now (2.7) follows directly from (2.8) and (2.9). This completes the proof. \square

From now on, fix $\{F_n : n \in \mathbb{N}\}$, a Følner sequence of G with the property that $e_G \subseteq F_1 \subsetneq F_2 \subsetneq \dots$ (it is easy to see that such a Følner sequence of G must exist).

We end this section with the following example, which highlights the difference between Proposition 2.3 and Proposition 2.8 for $G = \mathbb{Z}$ (compared to more general groups).

Example 2.9. *There exists a monotone non-negative \mathbb{Z} -invariant sub-additive function $f : \mathcal{F}_{\mathbb{Z}} \rightarrow \mathbb{R}$ (in particular, f satisfies the assumption of Proposition 2.8 and so the sequence $\{\frac{f(\{1, \dots, n\})}{n} : n \in \mathbb{N}\}$ converges) such that*

$$(2.10) \quad \lim_{n \rightarrow \infty} \frac{f(\{1, \dots, n\})}{n} > \inf_{E \in \mathcal{F}_{\mathbb{Z}}} \frac{f(E)}{|E|}.$$

Thus, f does not satisfy the assumption of Proposition 2.3.

Construction of Example 2.9. The function f is constructed as follows: let $E \in \mathcal{F}_{\mathbb{Z}}$,

$$f(E) = \min\{|E| - |F| : \{p + S : p \in F\} \text{ is a disjoint family of subsets of } E\},$$

here $S = \{1, 2, 4\}$ and F may be empty. For example, $f(S) = 2$, $f(\{1, 2, 3, 4\}) = 3$.

Now we claim that the constructed f has the required property.

First, we aim to prove that f is a monotone non-negative \mathbb{Z} -invariant sub-additive function by claiming $f(E) \leq f(E \cup \{a\})$ with $E \in \mathcal{F}_{\mathbb{Z}}$, $a \in \mathbb{Z} \setminus E$ and $f(E_1 \cup E_2) \leq f(E_1) + f(E_2)$ with $E_1, E_2 \in \mathcal{F}_{\mathbb{Z}}$, $E_1 \cap E_2 = \emptyset$.

Observe that we can select F such that $f(E \cup \{a\}) = |E| + 1 - |F|$ and $\{p + S : p \in F\}$ is a disjoint family of subsets of $E \cup \{a\}$. If $a \notin \cup\{p + S : p \in F\}$ then $\{p + S : p \in F\}$ is also a disjoint family of subsets of E and so $f(E) \leq |E| - |F|$. If $a \in p_0 + S$ for some $p_0 \in F$ then $\{p + S : p \in F \setminus \{p_0\}\}$ is a disjoint family of subsets of E and so $f(E) \leq |E| - |F \setminus \{p_0\}|$. Summing up, $f(E) \leq f(E \cup \{a\})$.

Now let F_i be such that $f(E_i) = |E_i| - |F_i|$ and $\{p + S : p \in F_i\}$ is a disjoint family of subsets of E_i , $i = 1, 2$. As $E_1 \cap E_2 = \emptyset$ It is easy to see that $F_1 \cap F_2 = \emptyset$ and $\{p + S : p \in F_1 \cup F_2\}$ is a disjoint family of subsets of $E_1 \cup E_2$, and so $f(E_1 \cup E_2) \leq |E_1 \cup E_2| - |F_1 \cup F_2| = f(E_1) + f(E_2)$.

Secondly, let $n \in \mathbb{N}$. We prove that $f(\{1, \dots, 4n\}) = 3n$. It is easy to check that $f(\{1, \dots, 4n\}) \leq 3n$. Assume that $f(\{1, \dots, 4n\}) < 3n$: in particular, there exists $F \in \mathcal{F}_{\mathbb{Z}}$ such that $\{p + S : p \in F\}$ is a disjoint family of subsets of $\{1, \dots, 4n\}$ and $|F| > n$. Observe that there exists at least one k such that $\{4k - 3, 4k - 2, 4k - 1, 4k\} \cap F$ contains at least two different elements. In particular, there exists $i', j' \in \{4k - 3, 4k - 2, 4k - 1, 4k\}$ such that $i' + S$ and $j' + S$ are disjoint, a contradiction to the fact that $(i + S) \cap (j + S) \neq \emptyset$ whenever $i, j \in \{1, 2, 3, 4\}$ (this can be verified directly). Thus, $f(\{1, \dots, 4n\}) = 3n$.

Finally, we finish the proof of the strict inequality (2.10) by observing that $\inf_{E \in \mathcal{F}_{\mathbb{Z}}} \frac{f(E)}{|E|} = \frac{2}{3}$. This finishes the construction. \square

Obviously, by standard modifications, we could obtain such an example with

$$\lim_{n \rightarrow \infty} \frac{f(\{1, \dots, n\})}{n} > 0 = \inf_{E \in \mathcal{F}_{\mathbb{Z}}} \frac{f(E)}{|E|}.$$

3. MEASURABLE DYNAMICAL SYSTEMS

In this section we give some background on measurable dynamical systems and obtain the relative Pinsker formula for an MDS for an infinite countable discrete amenable group action. This was obtained in [30] in the case where X is a Lebesgue space.

We believe that Theorem 3.13 is an interesting new result. Answering the related Question 3.14 will increase our understanding of the entropy theory of an MDS.

Let (Y, \mathcal{D}, ν) be a probability space. A *cover* of (Y, \mathcal{D}, ν) is a family $\mathcal{W} \subseteq \mathcal{D}$ satisfying $\bigcup_{W \in \mathcal{W}} W = Y$; if all elements of a cover \mathcal{W} are disjoint, then \mathcal{W} is called a *partition* of (Y, \mathcal{D}, ν) . Denote by \mathbf{C}_Y and \mathbf{P}_Y the set of all finite covers and finite partitions of (Y, \mathcal{D}, ν) , respectively. Let $\alpha \in \mathbf{P}_Y$ and $y \in Y$. Denote by $\alpha(y)$ the atom of α containing y . Let $\mathcal{W}_1, \mathcal{W}_2 \in \mathbf{C}_Y$. If each element of \mathcal{W}_1 is contained in some element of \mathcal{W}_2 then we say that \mathcal{W}_1 is *finer than* \mathcal{W}_2 (denote by $\mathcal{W}_1 \succeq \mathcal{W}_2$ or $\mathcal{W}_2 \preceq \mathcal{W}_1$). The join $\mathcal{W}_1 \vee \mathcal{W}_2$ of \mathcal{W}_1 and \mathcal{W}_2 is given by

$$\mathcal{W}_1 \vee \mathcal{W}_2 = \{W_1 \cap W_2 : W_1 \in \mathcal{W}_1, W_2 \in \mathcal{W}_2\}.$$

The definition extends naturally to finite collections of covers.

Fix $\mathcal{W}_1 \in \mathbf{C}_Y$ and denote by $\mathcal{P}(\mathcal{W}_1) \in \mathbf{P}_Y$ the finite partition generated by \mathcal{W}_1 : that is, if we say $\mathcal{W}_1 = \{W_1^1, \dots, W_1^m\}$, $m \in \mathbb{N}$ then

$$\mathcal{P}(\mathcal{W}_1) = \left\{ \bigcap_{i=1}^m A_i : A_i \in \{W_1^i, (W_1^i)^c\}, 1 \leq i \leq m \right\}.$$

We introduce a finite collection of partitions which we will use in the sequel. Let

$$\mathbf{P}(\mathcal{W}_1) = \{\alpha \in \mathbf{P}_Y : \mathcal{P}(\mathcal{W}_1) \succeq \alpha \succeq \mathcal{W}_1\}.$$

Now let \mathcal{C} be a sub- σ -algebra of \mathcal{D} and $\mathcal{W}_1 \in \mathbf{P}_Y$. We set

$$H_\nu(\mathcal{W}_1 | \mathcal{C}) = - \sum_{W_1 \in \mathcal{W}_1} \int_Y \nu(W_1 | \mathcal{C})(y) \log \nu(W_1 | \mathcal{C})(y) d\nu(y),$$

(by convention, we set $0 \log 0 = 0$). Here, $\nu(W_1 | \mathcal{C})$ denotes the conditional expectation with respect to ν of the function 1_{W_1} relative to \mathcal{C} . It is a standard fact that $H_\nu(\mathcal{W}_1 | \mathcal{C})$ increases with \mathcal{W}_1 (ordered by \succeq) and decreases as \mathcal{C} increases (ordered by \subseteq). In fact, if the sequence of sub- σ -algebras $\{\mathcal{C}_n : n \in \mathbb{N}\}$ increases or decreases to \mathcal{C} then the sequence $\{H_\nu(\mathcal{W}_1 | \mathcal{C}_n) : n \in \mathbb{N}\}$ decreases or increases to $H_\nu(\mathcal{W}_1 | \mathcal{C})$, respectively (see for example [29, Theorem 14.28]).

If $\mathcal{N}_Y \doteq \{\emptyset, Y\}$ is the trivial σ -algebra, one has

$$H_\nu(\mathcal{W}_1 | \mathcal{N}_Y) = - \sum_{W_1 \in \mathcal{W}_1} \nu(W_1) \log \nu(W_1) \geq H_\nu(\mathcal{W}_1 | \mathcal{C}).$$

We will write for short $H_\nu(\mathcal{W}_1) = H_\nu(\mathcal{W}_1 | \mathcal{N}_Y)$.

Let $\mathcal{W}_2 \in \mathbf{P}_Y$. Then \mathcal{W}_2 naturally generates a sub- σ -algebra of \mathcal{D} (also denoted by \mathcal{W}_2 if there is no ambiguity). It is easy to see that

$$H_\nu(\mathcal{W}_1 | \mathcal{W}_2) = H_\nu(\mathcal{W}_1 \vee \mathcal{W}_2) - H_\nu(\mathcal{W}_2).$$

In fact, more generally,

$$(3.1) \quad H_\nu(\mathcal{W}_1 | \mathcal{C} \vee \mathcal{W}_2) = H_\nu(\mathcal{W}_1 \vee \mathcal{W}_2 | \mathcal{C}) - H_\nu(\mathcal{W}_2 | \mathcal{C}),$$

here, $\mathcal{C} \vee \mathcal{W}_2$ denotes the sub- σ -algebra of \mathcal{D} generated by sub- σ -algebras \mathcal{C} and \mathcal{W}_2 (the notation works similarly for any given family of sub- σ -algebras of \mathcal{D}).

Now let $\mathcal{W}_1 \in \mathbf{C}_Y$, following the ideas of Romagnoli [63] we set

$$H_\nu(\mathcal{W}_1 | \mathcal{C}) = \inf_{\alpha \in \mathbf{P}_Y, \alpha \succeq \mathcal{W}_1} H_\nu(\alpha | \mathcal{C}).$$

Obviously, there is no ambiguity for this notation. Moreover, it remains true that $H_\nu(\mathcal{W}_1|\mathcal{C})$ increases with \mathcal{W}_1 and decreases as \mathcal{C} increases. Similarly, we can introduce $H_\nu(\mathcal{W}_1)$. Note that (see for example [63, Proposition 6])

$$(3.2) \quad H_\nu(\mathcal{W}_1) = \min_{\alpha \in \mathbf{P}(\mathcal{W}_1)} H_\nu(\alpha).$$

Let (Y, \mathcal{D}, ν, G) be an MDS, $\mathcal{W} \in \mathbf{C}_X$ and $\mathcal{C} \subseteq \mathcal{D}$ a sub- σ -algebra. For each $F \in \mathcal{F}_G$, set $\mathcal{W}_F = \bigvee_{g \in F} g^{-1}\mathcal{W}$. If \mathcal{C} is G -invariant, i.e. $g^{-1}\mathcal{C} = \mathcal{C}$ (up to ν null sets) for each $g \in G$, then it is easy to check that

$$H_\nu(\mathcal{W}_\bullet|\mathcal{C}) : \mathcal{F}_G \rightarrow \mathbb{R}, F \mapsto H_\nu(\mathcal{W}_F|\mathcal{C})$$

is a monotone non-negative G -invariant sub-additive function. Now, following Romagnoli [63] we may define the *measure-theoretic ν -entropy of \mathcal{W} with respect to \mathcal{C}* and the *measure-theoretic $\nu, +$ -entropy of \mathcal{W} with respect to \mathcal{C}* by

$$h_\nu(G, \mathcal{W}|\mathcal{C}) = \lim_{n \rightarrow \infty} \frac{1}{|F_n|} H_\nu(\mathcal{W}_{F_n}|\mathcal{C})$$

and

$$h_{\nu,+}(G, \mathcal{W}|\mathcal{C}) = \inf_{\alpha \in \mathbf{P}_Y, \alpha \succeq \mathcal{W}} h_\nu(G, \alpha|\mathcal{C}) \geq h_\nu(G, \mathcal{W}|\mathcal{C}),$$

respectively. By Proposition 2.2, $h_\nu(G, \mathcal{W}|\mathcal{C})$ and thus $h_{\nu,+}(G, \mathcal{W}|\mathcal{C})$ are well-defined. Observe that if $\alpha \in \mathbf{P}_Y$ then $h_\nu(G, \alpha|\mathcal{C}) = h_{\nu,+}(G, \alpha|\mathcal{C})$ and

$$(3.3) \quad h_\nu(G, \alpha|\mathcal{C}) = \inf_{F \in \mathcal{F}_G} \frac{1}{|F|} H_\nu(\alpha_F|\mathcal{C}) \leq H_\nu(\alpha|\mathcal{C}),$$

which is a direct corollary of Proposition 2.3, see also [20, (2)]. Then the *measure-theoretic ν -entropy of (Y, \mathcal{D}, ν, G) with respect to \mathcal{C}* is defined as

$$h_\nu(G, Y|\mathcal{C}) = \sup_{\alpha \in \mathbf{P}_Y} h_\nu(G, \alpha|\mathcal{C}).$$

By Proposition 2.2, all values of these invariants are independent of the selection of the Følner sequence $\{F_n : n \in \mathbb{N}\}$.

To simplify notation, when $\mathcal{C} = \mathcal{N}_Y$ we shall omit the qualification “with respect to \mathcal{C} ” or “ $|\mathcal{C}$ ”. When T is an invertible measure-preserving transformation of (Y, \mathcal{D}, ν) and we consider the group action of $\{T^n : n \in \mathbb{Z}\}$, we shall replace “ $\{T^n : n \in \mathbb{Z}\}$ ” by “ T ”.

It is not hard to obtain the following basic facts.

Proposition 3.1. *Let (Y, \mathcal{D}, ν, G) be an MDS, $\mathcal{W}_1, \mathcal{W}_2 \in \mathbf{C}_Y$, $\alpha_1, \alpha_2 \in \mathbf{P}_Y$, $F \in \mathcal{F}_G$ and $\mathcal{C} \subseteq \mathcal{D}$ a G -invariant sub- σ -algebra. Then*

- (1) $h_\nu(G, \mathcal{W}_1|\mathcal{C}) \leq h_\nu(G, \mathcal{W}_2|\mathcal{C})$ and $h_{\nu,+}(G, \mathcal{W}_1|\mathcal{C}) \leq h_{\nu,+}(G, \mathcal{W}_2|\mathcal{C})$ if $\mathcal{W}_1 \preceq \mathcal{W}_2$.
- (2) $h_\nu(G, \mathcal{W}_1 \vee \mathcal{W}_2|\mathcal{C}) \leq h_\nu(G, \mathcal{W}_1|\mathcal{C}) + h_\nu(G, \mathcal{W}_2|\mathcal{C})$ and $h_{\nu,+}(G, \mathcal{W}_1 \vee \mathcal{W}_2|\mathcal{C}) \leq h_{\nu,+}(G, \mathcal{W}_1|\mathcal{C}) + h_{\nu,+}(G, \mathcal{W}_2|\mathcal{C})$.
- (3) $h_\nu(G, (\mathcal{W}_1)_F|\mathcal{C}) = h_\nu(G, \mathcal{W}_1|\mathcal{C}) \leq h_{\nu,+}(G, \mathcal{W}_1|\mathcal{C}) \leq H_\nu(\mathcal{W}_1|\mathcal{C}) \leq \log |\mathcal{W}_1|$, here $|\mathcal{W}_1|$ denotes the cardinality of \mathcal{W}_1 .
- (4) $h_\nu(G, \alpha_1 \vee \alpha_2|\mathcal{C}) \leq h_\nu(G, \alpha_2|\mathcal{C}) + H_\nu(\alpha_1|\mathcal{C} \vee \alpha_2) \leq h_\nu(G, \alpha_2|\mathcal{C}) + H_\nu(\alpha_1|\alpha_2)$.
- (5) $h_\nu(G, Y|\mathcal{C}) = \sup_{\mathcal{W} \in \mathbf{C}_Y} h_\nu(G, \mathcal{W}|\mathcal{C}) = \sup_{\mathcal{W} \in \mathbf{C}_Y} h_{\nu,+}(G, \mathcal{W}|\mathcal{C})$.

Proof. Equations (1) and (5) are easy to verify.

Equations (2) and (4) follow directly from

$$H_\nu((\mathcal{W}_1 \vee \mathcal{W}_2)_E | \mathcal{C}) \leq H_\nu((\mathcal{W}_1)_E | \mathcal{C}) + H_\nu((\mathcal{W}_2)_E | \mathcal{C})$$

and

$$H_\nu((\alpha_1 \vee \alpha_2)_E | \mathcal{C}) \leq H_\nu((\alpha_2)_E | \mathcal{C}) + |E| H_\nu(\alpha_1 | \alpha_2 \vee \mathcal{C})$$

for each $E \in \mathcal{F}_G$, respectively, neither of which is hard to obtain.

Thus, we only need prove (3). Note that if $\alpha \in \mathbf{P}_Y$ satisfies $\alpha \succeq \mathcal{W}_1$ then

$$h_{\nu,+}(G, \mathcal{W}_1 | \mathcal{C}) \leq h_\nu(G, \alpha | \mathcal{C}) \leq H_\nu(\alpha | \mathcal{C})$$

by (3.3), which implies that

$$h_{\nu,+}(G, \mathcal{W}_1 | \mathcal{C}) \leq H_\nu(\mathcal{W}_1 | \mathcal{C}) \leq H_\nu(\mathcal{W}_1) \leq \log |\mathcal{W}_1|.$$

It remains to prove that

$$h_\nu(G, (\mathcal{W}_1)_F | \mathcal{C}) = h_\nu(G, \mathcal{W}_1 | \mathcal{C}).$$

We should point out that if $\{F_n : n \in \mathbb{N}\}$ is a Følner sequence of G then $\{FF_n : n \in \mathbb{N}\}$ is also a Følner sequence of G and $\lim_{n \rightarrow \infty} \frac{|FF_n|}{|F_n|} = 1$, which implies that

$$\begin{aligned} & h_\nu(G, (\mathcal{W}_1)_F | \mathcal{C}) \\ &= \lim_{n \rightarrow \infty} \frac{1}{|F_n|} H_\nu(((\mathcal{W}_1)_F)_{F_n} | \mathcal{C}) \\ &= \lim_{n \rightarrow \infty} \frac{1}{|F_n|} H_\nu((\mathcal{W}_1)_{FF_n} | \mathcal{C}) \\ &= \lim_{n \rightarrow \infty} \frac{1}{|FF_n|} H_\nu((\mathcal{W}_1)_{FF_n} | \mathcal{C}) \text{ (as } \lim_{n \rightarrow \infty} \frac{|FF_n|}{|F_n|} = 1) \\ &= h_\nu(G, \mathcal{W}_1 | \mathcal{C}) \text{ (as } \{FF_n : n \in \mathbb{N}\} \text{ is also a Følner sequence of } G). \end{aligned}$$

This proves (3) and so finishes our proof. \square

We also have:

Proposition 3.2. *Let (Y, \mathcal{D}, ν, G) be an MDS and $\mathcal{C} \subseteq \mathcal{D}$ a G -invariant sub- σ -algebra. Then for each $M \in \mathbb{N}$ and any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$|h_\nu(G, \mathcal{W}_1 | \mathcal{C}) - h_\nu(G, \mathcal{W}_2 | \mathcal{C})| < \epsilon$$

whenever $\mathcal{W}_j = \{W_{1,j}, \dots, W_{M,j}\} \in \mathbf{C}_Y, j = 1, 2$ satisfy $\sum_{m=1}^M \nu(W_{m,1} \Delta W_{m,2}) < \delta$.

Proof. This is just a re-writing of the proof of [37, Lemma 3.7]. \square

In fact, the following interesting result holds. This plays an important role in the establishment of the theory of local entropy theory for a topological G -action (see [37]).

Theorem 3.3. *Let (Y, \mathcal{D}, ν, G) be an MDS, $\mathcal{W} \in \mathbf{C}_Y$ and $\mathcal{C} \subseteq \mathcal{D}$ a G -invariant sub- σ -algebra. Assume that (Y, \mathcal{D}, ν) is a Lebesgue space. Then $h_\nu(G, \mathcal{W} | \mathcal{C}) = h_{\nu,+}(G, \mathcal{W} | \mathcal{C})$. Thus, using (3.3) we have an alternative expression for $h_\nu(G, \mathcal{W} | \mathcal{C})$:*

$$(3.4) \quad h_\nu(G, \mathcal{W} | \mathcal{C}) = \inf_{F \in \mathcal{F}_G} \frac{1}{|F|} \inf_{\alpha \in \mathbf{P}_Y, \alpha \succeq \mathcal{W}} H_\nu(\alpha_F | \mathcal{C}).$$

Remark 3.4. To prove Theorem 3.3, we shall use Danilenko's orbital approach to the entropy theory of an MDS as a crucial tool. In fact, to prove Theorem 3.3, we should recall almost all of the arguments of Danilenko in [16] and then re-write the whole process carried out in [37, §4]. In other words, we should argue the whole [37, §4] in the relative case of given a G -invariant sub- σ -algebra $\mathcal{C} \subseteq \mathcal{D}$. As this is a straightforward re-writing of the arguments of [37, §4], we shall omit the details and leave their verification to the interested reader. We only remark that, based on the results from [31, 33, 63], the equivalence of these two kinds of entropy for finite measurable covers was first pointed out in the literature by Huang, Ye and the second author of the paper in [35] in the case of \mathbb{Z} -actions.

As in the case of a measurable dynamical \mathbb{Z} -system, one can define a relative Pinsker formula in our setting.

Theorem 3.5. Let (Y, \mathcal{D}, ν, G) be an MDS, $\mathcal{C} \subseteq \mathcal{D}$ a G -invariant sub- σ -algebra and $\alpha, \beta \in \mathbf{P}_Y$. Then, for β_G , the sub- σ -algebra of \mathcal{D} generated by $g^{-1}\beta, g \in G$,

$$(3.5) \quad \lim_{n \rightarrow \infty} \frac{1}{|F_n|} H_\nu(\alpha_{F_n} | \beta_{F_n} \vee \mathcal{C}) = h_\nu(G, \alpha | \beta_G \vee \mathcal{C})$$

and so

$$h_\nu(G, \alpha \vee \beta | \mathcal{C}) = h_\nu(G, \beta | \mathcal{C}) + h_\nu(G, \alpha | \beta_G \vee \mathcal{C}).$$

Before establishing (3.5), we first make a remark.

Remark 3.6. Under the assumptions of Theorem 3.5, it is not hard to check that

$$H_\nu(\alpha_\bullet | \beta_\bullet \vee \mathcal{C}) : \mathcal{F}_G \rightarrow \mathbb{R}, F \mapsto H_\nu(\alpha_F | \beta_F \vee \mathcal{C})$$

is a non-negative G -invariant function. In fact, it is also sub-additive (using (3.1)):

$$\begin{aligned} H_\nu(\alpha_{E \cup F} | \beta_{E \cup F} \vee \mathcal{C}) &\leq H_\nu(\alpha_E | \beta_{E \cup F} \vee \mathcal{C}) + H_\nu(\alpha_F | \beta_{E \cup F} \vee \mathcal{C}) \\ &\leq H_\nu(\alpha_E | \beta_E \vee \mathcal{C}) + H_\nu(\alpha_F | \beta_F \vee \mathcal{C}) \end{aligned}$$

whenever $E, F \in \mathcal{F}_G$. In general, this function is not monotone. For example, let $G = \mathbb{Z}_2 \times \mathbb{Z}$ (hence $(0, 0)$ will be the unit of the group) and consider the MDS

$$(\{a, b\}^G, \mathcal{B}_{\{a, b\}^G}, \bigotimes_{g \in G} \{\frac{1}{2}, \frac{1}{2}\}, G),$$

where $\mathcal{B}_{\{a, b\}^G}$ denotes the Borel σ -algebra of the compact metric space $\{a, b\}^G$ and G acts naturally on $(\{a, b\}^G, \mathcal{B}_{\{a, b\}^G}, \bigotimes_{g \in G} \{\frac{1}{2}, \frac{1}{2}\})$ measure-preserving, set

$$\alpha = \{[a]_{(0,0)}, [b]_{(0,0)}\} \text{ and } \beta = (1, 0)^{-1} \alpha$$

with $[i]_{(0,0)} = \{(x_g)_{g \in G} : x_{(0,0)} = i\}$, $i \in \{a, b\}$. Now let $S \in \mathcal{F}_\mathbb{Z}$ and set

$$E = \{(0, s) : s \in S\} \in \mathcal{F}_G \text{ and } F = \{(1, s) : s \in S\} = (1, 0) \cdot E \in \mathcal{F}_G.$$

Using (3.1) again, it is straightforward to check

$$H_\nu(\alpha_F | \beta_F \vee \mathcal{N}_{\{a, b\}^G}) = H_\nu(\alpha_F \vee \beta_F | \mathcal{N}_{\{a, b\}^G}) - H_\nu(\beta_F | \mathcal{N}_{\{a, b\}^G}) = |S| \log 2;$$

whereas,

$$\alpha_F = \alpha_{(1,0) \cdot E} = ((1, 0)^{-1} \alpha)_E = \beta_E \text{ and similarly } \alpha_E = \beta_F,$$

and so

$$H_\nu(\alpha_{E \cup F} | \beta_{E \cup F} \vee \mathcal{N}_{\{a, b\}^G}) = 0 < |S| \log 2 = H_\nu(\alpha_F | \beta_F \vee \mathcal{N}_{\{a, b\}^G}).$$

Now we prove Theorem 3.5.

Proof of Theorem 3.5. Observe that for each $n \in \mathbb{N}$ using (3.1) one has

$$(3.6) \quad H_\nu((\alpha \vee \beta)_{F_n} | \mathcal{C}) = H_\nu(\beta_{F_n} | \mathcal{C}) + H_\nu(\alpha_{F_n} | \beta_{F_n} \vee \mathcal{C}).$$

By the definitions, to finish the proof it is sufficient to prove (3.5).

As a sub- σ -algebra of \mathcal{D} , β_G (and likewise $\beta_G \vee \mathcal{C}$) is G -invariant, thus

$$h_\nu(G, \alpha | \beta_G \vee \mathcal{C}) = \lim_{n \rightarrow \infty} \frac{1}{|F_n|} H_\nu(\alpha_{F_n} | \beta_G \vee \mathcal{C}).$$

Set $M = H_\nu(\alpha | \beta \vee \mathcal{C})$ (and so $M = H_\nu(\alpha_{\{g\}} | \beta_{\{g\}} \vee \mathcal{C})$ for each $g \in G$) and

$$c = \lim_{n \rightarrow \infty} \frac{1}{|F_n|} H_\nu(\alpha_{F_n} | \beta_{F_n} \vee \mathcal{C}).$$

Observe that by Proposition 2.2 the limit c must exist (using (3.6)).

Obviously, $c \geq h_\nu(G, \alpha | \beta_G \vee \mathcal{C})$. To complete the proof, we only need show that $c \leq h_\nu(G, \alpha | \beta_G \vee \mathcal{C})$. The proof follows from the methods of [68, Proposition 4.3].

Let $\epsilon \in (0, \frac{1}{4})$. Clearly, there exists $N \in \mathbb{N}$ such that if $n > N$ then

$$(3.7) \quad \left| \frac{1}{|F_n|} H_\nu(\alpha_{F_n} | \beta_{F_n} \vee \mathcal{C}) - c \right| < \epsilon \text{ and } \left| \frac{1}{|F_n|} H_\nu(\alpha_{F_n} | \beta_G \vee \mathcal{C}) - h_\nu(\alpha | \beta_G \vee \mathcal{C}) \right| < \epsilon.$$

By Proposition 2.1, there exist integers n_1, \dots, n_k such that $N \leq n_1 < \dots < n_k$ and F_{n_1}, \dots, F_{n_k} ϵ -quasi-tile F_m whenever m is sufficiently large. Note that there must exist $B \in \mathcal{F}_G$ such that

$$(3.8) \quad H_\nu(\alpha_{F_{n_i}} | \beta_B \vee \mathcal{C}) \leq H_\nu(\alpha_{F_{n_i}} | \beta_G \vee \mathcal{C}) + \epsilon$$

for each $i = 1, \dots, k$. Now let $m \in \mathbb{N}, m > N$ be large enough such that F_m is $(B \cup \{e_G\}, \frac{\epsilon}{\sum_{i=1}^k |F_{n_i}|})$ -invariant and F_{n_1}, \dots, F_{n_k} ϵ -quasi-tile F_m with tiling centers

C_1^m, \dots, C_k^m . Then, by the selection of C_1^m, \dots, C_k^m , one has

- (1) for $A_m \doteq \{g \in F_m : Bg \subseteq F_m\} = F_m \setminus B^{-1}(G \setminus F_m)$, as $F_m \setminus A_m \subseteq F_m \cap B^{-1}(G \setminus F_m)$ and F_m is $(B \cup \{e_G\}, \frac{\epsilon}{\sum_{i=1}^k |F_{n_i}|})$ -invariant, then

$$|F_m \setminus A_m| < \frac{\epsilon |F_m|}{\sum_{i=1}^k |F_{n_i}|};$$

- (2) $C_i^m \subseteq F_m, i = 1, \dots, k$ (as $e_G \subseteq F_1 \subseteq F_2 \subseteq \dots$) and

- (3) $F_m \supseteq \bigcup_{i=1}^k F_{n_i} C_i^m$ and $|\bigcup_{i=1}^k F_{n_i} C_i^m| \geq \max\{(1-\epsilon)|F_m|, (1-\epsilon) \sum_{i=1}^k |C_i^m| |F_{n_i}|\}$.

Moreover, we have

$$(3.9) \quad \begin{aligned} & \frac{1}{|F_m|} H_\nu(\alpha_{F_m} | \beta_{F_m} \vee \mathcal{C}) \\ & \leq \frac{1}{|F_m|} \{H_\nu(\alpha_{\bigcup_{i=1}^k F_{n_i} C_i^m} | \beta_{F_m} \vee \mathcal{C}) + H_\nu(\alpha_{F_m \setminus \bigcup_{i=1}^k F_{n_i} C_i^m} | \mathcal{C})\} \\ & \leq \frac{1}{(1-\epsilon) \sum_{i=1}^k |C_i^m| |F_{n_i}|} \sum_{i=1}^k H_\nu(\alpha_{F_{n_i} C_i^m} | \beta_{F_m} \vee \mathcal{C}) + \epsilon \log |\alpha|, \end{aligned}$$

where the last inequality follows from the above (3), moreover, for each $i = 1, \dots, k$,

$$\begin{aligned}
& \frac{1}{|C_i^m||F_{n_i}|} H_\nu(\alpha_{F_{n_i} C_i^m} | \beta_{F_m} \vee \mathcal{C}) \\
& \leq \frac{1}{|C_i^m|} \sum_{c \in C_i^m} \frac{1}{|F_{n_i}|} H_\nu(\alpha_{F_{n_i} c} | \beta_{F_m} \vee \mathcal{C}) \\
& = \frac{1}{|C_i^m|} \sum_{c \in C_i^m} \frac{1}{|F_{n_i}|} H_\nu(\alpha_{F_{n_i}} | \beta_{F_m c^{-1}} \vee \mathcal{C}) \\
& \leq \frac{1}{|C_i^m|} \left\{ \sum_{c \in C_i^m \cap A_m} \frac{1}{|F_{n_i}|} H_\nu(\alpha_{F_{n_i}} | \beta_{F_m c^{-1}} \vee \mathcal{C}) + \right. \\
& \quad \left. \sum_{c \in C_i^m \setminus A_m} \frac{1}{|F_{n_i}|} H_\nu(\alpha_{F_{n_i}} | \beta_{F_m c^{-1}} \vee \mathcal{C}) \right\} \\
& \leq \frac{1}{|F_{n_i}|} H_\nu(\alpha_{F_{n_i}} | \beta_B \vee \mathcal{C}) + \frac{1}{|C_i^m|} \sum_{c \in F_m \setminus A_m} \frac{1}{|F_{n_i}|} H_\nu(\alpha_{F_{n_i}} | \beta_{F_m c^{-1}} \vee \mathcal{C}) \\
& \quad \text{(by the selection of } A_m \text{ and the above (2))} \\
(3.10) \quad & \leq \frac{1}{|F_{n_i}|} H_\nu(\alpha_{F_{n_i}} | \beta_G \vee \mathcal{C}) + \epsilon + \frac{|F_m \setminus A_m|}{|C_i^m|} \log |\alpha| \text{ (using (3.8)).}
\end{aligned}$$

Combining (3.9) and (3.10), we obtain

$$\begin{aligned}
& \frac{1}{|F_m|} H_\nu(\alpha_{F_m} | \beta_{F_m} \vee \mathcal{C}) \\
& \leq \frac{1}{1-\epsilon} \sum_{i=1}^k \frac{|C_i^m||F_{n_i}|}{\sum_{j=1}^k |C_j^m||F_{n_j}|} \left\{ \frac{1}{|F_{n_i}|} H_\nu(\alpha_{F_{n_i}} | \beta_G \vee \mathcal{C}) + \right. \\
& \quad \left. \epsilon + \frac{|F_m \setminus A_m|}{|C_i^m|} \log |\alpha| \right\} + \epsilon \log |\alpha| \\
& \leq \frac{1}{1-\epsilon} \left\{ \max_{1 \leq i \leq k} \frac{1}{|F_{n_i}|} H_\nu(\alpha_{F_{n_i}} | \beta_G \vee \mathcal{C}) + \right. \\
& \quad \left. \epsilon + \frac{\epsilon |F_m|}{\sum_{i=1}^k |C_i^m||F_{n_i}|} \log |\alpha| \right\} + \epsilon \log |\alpha| \text{ (using (1))} \\
& \leq \frac{1}{1-\epsilon} \max_{1 \leq i \leq k} \frac{1}{|F_{n_i}|} H_\nu(\alpha_{F_{n_i}} | \beta_G \vee \mathcal{C}) + \\
& \quad \frac{1}{1-\epsilon} \left(\epsilon + \frac{\epsilon}{1-\epsilon} \log |\alpha| \right) + \epsilon \log |\alpha| \text{ (using (3))},
\end{aligned}$$

combined with (3.7), one has

$$c < \frac{1}{1-\epsilon} h_\nu(G, \alpha | \beta_G \vee \mathcal{C}) + \frac{1}{1-\epsilon} (2\epsilon + \frac{\epsilon}{1-\epsilon} \log |\alpha|) + \epsilon(1 + \log |\alpha|).$$

Finally, $c \leq h_\nu(G, \alpha | \beta_G \vee \mathcal{C})$ follows by letting $\epsilon \rightarrow 0$. This finishes our proof. \square

Remark 3.7. Remark that the case where (Y, \mathcal{D}, ν) is a Lebesgue space was proved by Glasner, Thouvenot and Weiss [30, Lemma 1.1]. The relative Pinsker formula for a measurable dynamical \mathbb{Z} -system is proved in [72, Theorem 3.3].

Let (Y, \mathcal{D}, ν) be a Lebesgue space. If $\{\alpha_i : i \in I\}$ is a countable family in \mathbf{P}_Y , the partition $\alpha = \bigvee_{i \in I} \alpha_i \doteq \{\bigcap_{i \in I} A_i : A_i \in \alpha_i, i \in I\}$ is called a *measurable partition*. Note that the sets $C \in \mathcal{D}$, which are unions of atoms of α , form a sub- σ -algebra of \mathcal{D} , which we will also denote by α without any ambiguity. In fact, every sub- σ -algebra of \mathcal{D} coincides with a σ -algebra constructed in this way modulo ν -null sets (cf [61]).

Let (Y, \mathcal{D}, ν, G) be an MDS and $\mathcal{C} \subseteq \mathcal{D}$ a G -invariant sub- σ -algebra. Define the *Pinsker algebra of (Y, \mathcal{D}, ν, G) with respect to \mathcal{C}* , $\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)$, to be the sub- σ -algebra of \mathcal{D} generated by $\{\alpha \in \mathbf{P}_Y : h_\nu(G, \alpha|\mathcal{C}) = 0\}$. In the case of $\mathcal{C} = \mathcal{N}_Y$ we will write $\mathcal{P}(Y, \mathcal{D}, \nu, G) = \mathcal{P}^{\mathcal{N}_Y}(Y, \mathcal{D}, \nu, G)$ and call it the *Pinsker algebra of (Y, \mathcal{D}, ν, G)* . Obviously $\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G) \subseteq \mathcal{D}$ is a G -invariant sub- σ -algebra and $\mathcal{C} \cup \mathcal{P}(Y, \mathcal{D}, \nu, G) \subseteq \mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)$.

We say that (Y, \mathcal{D}, ν, G) has *\mathcal{C} -relative c.p.e.* if $\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G) = \mathcal{C}$ (in the sense of mod ν), and has *c.p.e.* if it has \mathcal{N}_Y -relative c.p.e.

The following is [20, Theorem 3.1].

Proposition 3.8. *Let (Y, \mathcal{D}, ν, G) be an MDS and $\mathcal{C} \subseteq \mathcal{D}$ a G -invariant sub- σ -algebra. Assume that (Y, \mathcal{D}, ν) is a Lebesgue space. Then (Y, \mathcal{D}, ν, G) has \mathcal{C} -relative c.p.e. if and only if for each $\alpha \in \mathbf{P}_Y$ and any $\epsilon > 0$ there exists $K \in \mathcal{F}_G$ such that if $F \in \mathcal{F}_G$ satisfies $FF^{-1} \cap (K \setminus \{e_G\}) = \emptyset$ then*

$$\left| \frac{1}{|F|} H_\nu(\alpha_F|\mathcal{C}) - H_\nu(\alpha|\mathcal{C}) \right| < \epsilon.$$

We also have:

Proposition 3.9. *Let (Y, \mathcal{D}, ν, G) be an MDS, $\mathcal{C} \subseteq \mathcal{D}$ a G -invariant sub- σ -algebra and $\alpha \in \mathbf{P}_Y$. Assume that (Y, \mathcal{D}, ν) is a Lebesgue space. Then*

$$(3.11) \quad h_\nu(G, \alpha|\mathcal{C}) = h_\nu(G, \alpha|\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)).$$

In particular, (Y, \mathcal{D}, ν, G) has $\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)$ -relative c.p.e.

Proof. First, let us prove (3.11). As (Y, \mathcal{D}, ν) is a Lebesgue space, there exists a sequence $\{\beta_n : n \in \mathbb{N}\} \subseteq \mathbf{P}_Y$ satisfying $\beta_1 \preceq \beta_2 \preceq \dots \nearrow \mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)$. For each $n \in \mathbb{N}$, one has

$$\begin{aligned} h_\nu(G, \alpha|\mathcal{C}) &\leq h_\nu(G, \alpha \vee \beta_n|\mathcal{C}) \\ &= h_\nu(G, \beta_n|\mathcal{C}) + h_\nu(G, \alpha|(\beta_n)_G \vee \mathcal{C}) \quad (\text{using Theorem 3.5}) \\ (3.12) \quad &= h_\nu(G, \alpha|(\beta_n)_G \vee \mathcal{C}) \quad (\text{as } \beta_n \subseteq \mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) \leq h_\nu(G, \alpha|\mathcal{C}). \end{aligned}$$

By the choice of the sequence $\{\beta_n : n \in \mathbb{N}\}$, the sequence of sub- σ -algebras $(\beta_n)_G \vee \mathcal{C}$ increases to $\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)$, and so by (3.12) one has:

$$\begin{aligned} h_\nu(G, \alpha|\mathcal{C}) &= \inf_{n \in \mathbb{N}} h_\nu(G, \alpha|(\beta_n)_G \vee \mathcal{C}) \\ &= \inf_{n \in \mathbb{N}} \inf_{F \in \mathcal{F}_G} \frac{1}{|F|} H_\nu(\alpha_F|(\beta_n)_G \vee \mathcal{C}) \quad (\text{using (3.3)}) \\ &= \inf_{F \in \mathcal{F}_G} \inf_{n \in \mathbb{N}} \frac{1}{|F|} H_\nu(\alpha_F|(\beta_n)_G \vee \mathcal{C}) \\ &= \inf_{F \in \mathcal{F}_G} \frac{1}{|F|} H_\nu(\alpha_F|\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) \\ (3.13) \quad &= h_\nu(G, \alpha|\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) \quad (\text{using (3.3) again}). \end{aligned}$$

This establishes (3.11). Moreover, from this one sees

$$\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G) = \mathcal{P}^{\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)}(Y, \mathcal{D}, \nu, G),$$

that is, (Y, \mathcal{D}, ν, G) has $\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)$ -relative c.p.e. This finishes our proof. \square

As another direct corollary of Theorem 3.5, one obtains the well-known Abramov-Rokhlin entropy addition formula (see for example [16, Theorem 0.2] or [68]).

Proposition 3.10. *Let (Y, \mathcal{D}, ν, G) be an MDS and $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{D}$ two G -invariant sub- σ -algebras. Assume that (Y, \mathcal{D}, ν) is a Lebesgue space. Then*

$$h_{\nu}(G, Y|\mathcal{C}_1) = h_{\nu}(G, Y|\mathcal{C}_2) + h_{\nu}(G, Y, \mathcal{C}_2|\mathcal{C}_1).$$

Here, $h_{\nu}(G, Y, \mathcal{C}_2|\mathcal{C}_1)$ denotes the measure-theoretic ν -entropy of the MDS $(Y, \mathcal{C}_2, \nu, G)$ with respect to \mathcal{C}_1 .

Proof. Let $\{\alpha_i : i \in \mathbb{N}\}$ and $\{\beta_i : i \in \mathbb{N}\}$ be two countable families in \mathbf{P}_Y such that the sub- σ -algebras \mathcal{C}_2 and \mathcal{D} can be induced by the measurable partitions $\bigvee_{i \in \mathbb{N}} \alpha_i$ and $\bigvee_{i \in \mathbb{N}} \beta_i$, respectively. By a similar reasoning to (3.13) one has

$$(3.14) \quad h_{\nu}(G, Y|\mathcal{C}_1) = \lim_{n \rightarrow \infty} h_{\nu}(G, \bigvee_{i=1}^n (\alpha_i \vee \beta_i)|\mathcal{C}_1),$$

$$(3.15) \quad h_{\nu}(G, Y|\mathcal{C}_2) = \lim_{n \rightarrow \infty} h_{\nu}(G, \bigvee_{i=1}^n \beta_i | (\bigvee_{i=1}^n \alpha_i)_G \vee \mathcal{C}_1) \text{ (as } \mathcal{C}_1 \subseteq \mathcal{C}_2),$$

$$(3.16) \quad h_{\nu}(G, Y, \mathcal{C}_2|\mathcal{C}_1) = \lim_{n \rightarrow \infty} h_{\nu}(G, \bigvee_{i=1}^n \alpha_i|\mathcal{C}_1).$$

For each $n \in \mathbb{N}$, by Theorem 3.5 one has

$$(3.17) \quad h_{\nu}(G, \bigvee_{i=1}^n (\alpha_i \vee \beta_i)|\mathcal{C}_1) = h_{\nu}(G, \bigvee_{i=1}^n \alpha_i|\mathcal{C}_1) + h_{\nu}(G, \bigvee_{i=1}^n \beta_i | (\bigvee_{i=1}^n \alpha_i)_G \vee \mathcal{C}_1).$$

The conclusion now follows from (3.14), (3.15), (3.16) and (3.17). \square

Let (Y, \mathcal{D}, ν, G) be an MDS and $\mathcal{C} \subseteq \mathcal{D}$ a G -invariant sub- σ -algebra. For each $n \in \mathbb{N} \setminus \{1\}$, over (Y^n, \mathcal{D}^n) (here, $Y^n = Y \times \cdots \times Y$ (n -times) and $\mathcal{D}^n = \mathcal{D} \times \cdots \times \mathcal{D}$ (n -times)) following ideas from [28, 34, 36, 37], we introduce a probability measure $\lambda_n^{\mathcal{C}}(\nu)$ as follows:

$$\lambda_n^{\mathcal{C}}(\nu)(\prod_{i=1}^n A_i) = \int_Y \prod_{i=1}^n \nu(A_i | \mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) d\nu,$$

whenever $A_1, \dots, A_n \in \mathcal{D}$. As G acts naturally on (Y^n, \mathcal{D}^n) , it is not hard to check that the measure $\lambda_n^{\mathcal{C}}(\nu)$ is G -invariant (recall that the sub- σ -algebra $\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G) \subseteq \mathcal{D}$ is G -invariant) and so $(Y^n, \mathcal{D}^n, \lambda_n^{\mathcal{C}}(\nu), G)$ forms an MDS.

Following the method of proof of [37, Lemma 6.8 and Theorem 6.11], it is not hard to obtain:

Lemma 3.11. *Let (Y, \mathcal{D}, ν, G) be an MDS, $\mathcal{C} \subseteq \mathcal{D}$ a G -invariant sub- σ -algebra and $\mathcal{W} = \{W_1, \dots, W_n\} \in \mathbf{C}_Y$ with $n \in \mathbb{N} \setminus \{1\}$. Then*

- (1) $\lambda_n^{\mathcal{C}}(\nu)(\prod_{i=1}^n W_i^c) > 0$ if and only if $h_\nu(G, \beta|\mathcal{C}) > 0$ whenever $\beta \in \mathbf{P}_Y$ satisfies $\beta \succeq \mathcal{W}$.
- (2) if $\lambda_n^{\mathcal{C}}(\nu)(\prod_{i=1}^n W_i^c) > 0$ then there exist $\epsilon > 0$ and $\alpha \in \mathbf{P}_Y$ such that $\alpha \succeq \mathcal{W}$ and, whenever $\beta \in \mathbf{P}_Y$ satisfies $\beta \succeq \mathcal{W}$,

$$H_\nu(\alpha|\beta \vee \mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) \leq H_\nu(\alpha|\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) - \epsilon.$$

Remark 3.12. In fact, following the proof of [37, Theorem 6.11], the partition $\alpha \in \mathbf{P}_Y$ in Lemma 3.11 (2) can be specified as follows: put $W_i(0) = W_i$ and $W_i(1) = W_i^c$ for each $i = 1, \dots, n$, set $\mathcal{W}_s = \bigcap_{i=1}^n W_i(s_i)$ for each $s = (s_1, \dots, s_n) \in \{0, 1\}^n$ and then define $\alpha = \{\mathcal{W}_s : s \in \{0, 1\}^n\}$.

This result can be strengthened as follows.

Theorem 3.13. Let (Y, \mathcal{D}, ν, G) be an MDS, $\mathcal{C} \subseteq \mathcal{D}$ a G -invariant sub- σ -algebra and $\mathcal{W} = \{W_1, \dots, W_n\} \in \mathbf{C}_Y$ with $n \in \mathbb{N} \setminus \{1\}$. Assume that (Y, \mathcal{D}, ν) is a Lebesgue space. Then the following statements are equivalent:

- (1) $h_\nu(G, \beta|\mathcal{C}) > 0$ whenever $\beta \in \mathbf{P}_Y$ satisfies $\beta \succeq \mathcal{W}$.
- (2) $\lambda_n^{\mathcal{C}}(\nu)(\prod_{i=1}^n W_i^c) > 0$.
- (3) $\inf_{F \in \mathcal{F}_G} \frac{1}{|F|} H_\nu(\mathcal{W}_F|\mathcal{C}) > 0$.
- (4) $h_\nu(G, \mathcal{W}|\mathcal{C}) > 0$.

Proof. The equivalence (1) \iff (2) is established by Lemma 3.11 and the implications (3) \implies (4) \implies (1) follow directly from the definitions.

Thus, it suffices to prove (2) \implies (3).

Now assume that $\lambda_n^{\mathcal{C}}(\nu)(\prod_{i=1}^n W_i^c) > 0$. Using Lemma 3.11 again, there exist $\alpha \in \mathbf{P}_Y$ and $\epsilon > 0$ such that

$$(3.18) \quad H_\nu(\alpha|\beta \vee \mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) \leq H_\nu(\alpha|\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) - \epsilon$$

whenever $\beta \in \mathbf{P}_Y$ satisfies $\beta \succeq \mathcal{W}$. By Proposition 3.8 and Proposition 3.9, we can choose $K \in \mathcal{F}_G$ such that if $F \in \mathcal{F}_G$ satisfies $FF^{-1} \cap (K \setminus \{e_G\}) = \emptyset$ then

$$(3.19) \quad \left| \frac{1}{|F|} H_\nu(\alpha_F|\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) - H_\nu(\alpha|\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) \right| < \frac{\epsilon}{2}.$$

For $E \in \mathcal{F}_G$ and $g \in E$, there exists $S \in \mathcal{F}_G$ such that $SS^{-1} \cap (K \setminus \{e_G\}) = \emptyset$, $g \in S \subseteq E$ and $(S \cup \{g'\})(S \cup \{g'\})^{-1} \cap (K \setminus \{e_G\}) \neq \emptyset$ for any $g' \in E \setminus S$. Thus,

$$(3.20) \quad \left| \frac{1}{|S|} H_\nu(\alpha_S|\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) - H_\nu(\alpha|\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) \right| < \frac{\epsilon}{2} \text{ (using (3.19)).}$$

It is now not hard to check that

$$E \setminus S \subseteq (K \setminus \{e_G\})S \cup (K \setminus \{e_G\})^{-1}S = (K \cup K^{-1} \setminus \{e_G\})S,$$

hence $S \subseteq E \subseteq (K \cup K^{-1} \cup \{e_G\})S$, one has $(2|K| + 1)|S| \geq |E|$. So, if $\beta \in \mathbf{P}_Y$ satisfies $\beta \succeq \mathcal{W}_S$ then $g\beta \succeq \mathcal{W}$ for each $g \in S$, hence

$$\begin{aligned}
& H_\nu(\beta|\mathcal{C}) \\
& \geq H_\nu(\beta|\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) \\
& = H_\nu(\beta \vee \alpha_S|\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) - H_\nu(\alpha_S|\beta \vee \mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) \\
& \geq H_\nu(\alpha_S|\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) - \sum_{g \in S} H_\nu(\alpha|g\beta \vee \mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) \\
& \geq H_\nu(\alpha_S|\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) - |S|(H_\nu(\alpha|\mathcal{P}^{\mathcal{C}}(Y, \mathcal{D}, \nu, G)) - \epsilon) \text{ (using (3.18))} \\
& \geq \frac{|S|\epsilon}{2} \text{ (using (3.20)).}
\end{aligned}$$

Since β is arbitrary,

$$H_\nu(\mathcal{W}_E) \geq H_\nu(\mathcal{W}_S) \geq \frac{|S|\epsilon}{2}.$$

Finally, letting E vary over all elements from \mathcal{F}_G we obtain (3). (Recall that $(2|K| + 1)|S| \geq |E|$.) This completes the proof. \square

Question 3.14. Let (Y, \mathcal{D}, ν, G) be an MDS, $\mathcal{C} \subseteq \mathcal{D}$ a G -invariant sub- σ -algebra and $\mathcal{W} \in \mathbf{C}_Y$. We conjecture that the following equation holds:

$$h_\nu(G, \mathcal{W}|\mathcal{C}) = \inf_{F \in \mathcal{F}_G} \frac{1}{|F|} H_\nu(\mathcal{W}_F|\mathcal{C}).$$

- (1) The reasoning of (3.3) does not work in this case, since if $\alpha \in \mathbf{P}_Y$ then using (3.1) one can demonstrate easily the strong sub-additivity of

$$(3.21) \quad H_\nu(\alpha_{E \cap F}|\mathcal{C}) + H_\nu(\alpha_{E \cup F}|\mathcal{C}) \leq H_\nu(\alpha_E|\mathcal{C}) + H_\nu(\alpha_F|\mathcal{C})$$

whenever $E, F \in \mathcal{F}_G$ (setting $\alpha_\emptyset = \mathcal{N}_Y$). We don't know whether (3.21) holds for a general cover $\mathcal{W} \in \mathbf{C}_Y$.

- (2) From the definitions, the inequality \geq holds directly. Moreover, by Theorem 3.13, if (Y, \mathcal{D}, ν) is a Lebesgue space then

$$\inf_{F \in \mathcal{F}_G} \frac{1}{|F|} H_\nu(\mathcal{W}_F|\mathcal{C}) > 0 \text{ if and only if } h_\nu(G, \mathcal{W}|\mathcal{C}) > 0.$$

- (3) The conjecture should be compared with Proposition 2.3, Proposition 2.8 and Example 2.9.

Observe that in the topological setting, we have a similar result [20, Lemma 6.1], and so a similar conjecture can be made.

Let (Y, \mathcal{D}, ν) be a Lebesgue space and $\mathcal{C} \subseteq \mathcal{D}$ a sub- σ -algebra. Then we may disintegrate ν over \mathcal{C} , i.e. we write $\nu = \int_Y \nu_y d\nu(y)$, where ν_y is a probability measure over (Y, \mathcal{D}) for ν -a.e. $y \in Y$. In fact, if α is a measurable partition of (Y, \mathcal{D}, ν) which generates \mathcal{C} , then, for ν -a.e. $y \in Y$, ν_y is supported on $\alpha(y)$ (i.e. $\nu_y(\alpha(y)) = 1$) and $\nu_{y_1} = \nu_{y_2}$ for ν_y -a.e. $y_1, y_2 \in \alpha(y)$. The disintegration can be characterized as follows: for each $f \in L^1(Y, \mathcal{D}, \nu)$, if we denote by $\nu(f|\mathcal{C})$ the conditional expectation with respect to ν of the function f relative to \mathcal{C} , then

- (1) $f \in L^1(Y, \mathcal{D}, \nu_y)$ for ν -a.e. $y \in Y$,
- (2) the function $y \mapsto \int_Y f d\nu_y$ is in $L^1(Y, \mathcal{C}, \nu)$ and
- (3) $\nu(f|\mathcal{C})(y) = \int_Y f d\nu_y$ for ν -a.e. $y \in Y$.

From this, it follows that if $f \in L^1(Y, \mathcal{D}, \nu)$ then

$$(3.22) \quad \int_Y \left(\int_Y f d\nu_y \right) d\nu(y) = \int_Y f d\nu,$$

and so it is simple to check that if $\beta \in \mathbf{P}_Y$ then

$$(3.23) \quad H_\nu(\beta|\mathcal{C}) = \int_Y H_{\nu_y}(\beta) d\nu(y).$$

Note that the disintegration is unique in the sense that if $\nu = \int_Y \nu_y d\nu(y)$ and $\nu = \int_Y \nu'_y d\nu(y)$ are both the disintegrations of ν over \mathcal{C} , then $\nu_y = \nu'_y$ for ν -a.e. $y \in Y$. For details see for example [27, 61].

Now let (Y, \mathcal{D}, ν, G) be an MDS and $\mathcal{C} \subseteq \mathcal{D}$ a G -invariant sub- σ -algebra. Assume that (Y, \mathcal{D}, ν) is a Lebesgue space and $\nu = \int_Y \nu_y d\nu(y)$ is the disintegration of ν over \mathcal{C} . Then, for each $n \in \mathbb{N} \setminus \{1\}$, by the construction of $\lambda_n^{\mathcal{C}}(\nu)$ one has:

$$\lambda_n^{\mathcal{C}}(\nu) = \int_Y \nu_y \times \cdots \times \nu_y \text{ (n-times) } d\nu(y).$$

As in [36, Lemma 3.8], we have:

Lemma 3.15. *Let (Y, \mathcal{D}, ν) be a Lebesgue space and $\mathcal{W} \in \mathbf{C}_Y$. Let $\mathcal{C} \subseteq \mathcal{D}$ be a sub- σ -algebra and $\nu = \int_Y \nu_y d\nu(y)$ the disintegration of ν over \mathcal{C} . Then*

$$H_\nu(\mathcal{W}|\mathcal{C}) = \int_Y H_{\nu_y}(\mathcal{W}) d\nu(y).$$

A probability space (Y, \mathcal{D}, ν) is called *purely atomic* if there exists a countably family $\{D_i : i \in I\} \subseteq \mathcal{D}$ such that $\nu(\bigcup_{i \in I} D_i) = 1$ and for each $i \in I$, $\nu(D_i) > 0$ and if $D'_i \subseteq D_i$ is measurable then $\nu(D'_i)$ is either 0 or $\nu(D_i)$.

We have (observe that [53, Theorem 1.1] is just a special case of Proposition 3.16):

Proposition 3.16. *Let (Y, \mathcal{D}, ν, G) be an MDS and $\mathcal{C} \subseteq \mathcal{D}$ a G -invariant sub- σ -algebra. Assume that (Y, \mathcal{D}, ν) is a Lebesgue space and $\nu = \int_Y \nu_y d\nu(y)$ is the disintegration of ν over \mathcal{C} . If ν_y is purely atomic for ν -a.e. $y \in Y$ then $h_\nu(G, Y|\mathcal{C}) = 0$. Conversely, if $h_\nu(G, Y|\mathcal{C}) > 0$ then there is $A \in \mathcal{D}$ such that $\nu(A) > 0$ and ν_y is not purely atomic for each $y \in A$.*

Remark 3.17. *The assumption that (Y, \mathcal{D}, ν) is a Lebesgue space in Lemma 3.15 and Proposition 3.16 is not essential. In fact, the conclusion holds whenever there is a disintegration of ν over the sub- σ -algebra $\mathcal{C} \subseteq \mathcal{D}$.*

The case where ν is ergodic in Proposition 3.16 is well known and is quite standard in ergodic theory (see [22, Theorem 4.1.15] for a stronger version). In fact, it is not hard to obtain Proposition 3.16 in the general case: based on the following result, we can prove it by standard arguments.

Lemma 3.18. *Let (X, \mathcal{B}, μ) be a purely atomic probability space and $\{\alpha_j : j \in \mathbb{N}\} \subseteq \mathbf{P}_X$. Then*

$$(3.24) \quad \left\{ x \in X : \lim_{n \rightarrow +\infty} \frac{1}{n} \log \mu \left(\bigcap_{j=1}^n \alpha_j(x) \right) = 0 \right\} \text{ has } \mu\text{-measure } 1,$$

and if there is $k > 0$ such that $|\alpha_j| \leq k$ for every j then in addition

$$(3.25) \quad \lim_{n \rightarrow \infty} \frac{1}{n} H_\mu \left(\bigvee_{j=1}^n \alpha_j \right) = 0.$$

Proof. By assumption, there exists a partition $\{B\} \cup \{B_i : i \in I\}$ of (X, \mathcal{B}, μ) with $I \subseteq \mathbb{N}$, such that $\mu(B) = 0$, $\mu(B_i) > 0$ for each $i \in I$ and

$$\mu\left(\bigcup_{i \in I} B_i\right) = 1 \text{ and } \bigcap_{j \in \mathbb{N}} \alpha_j(x) = B_i \text{ for each } x \in B_i.$$

Observe that for any $i \in I$ and $x \in B_i$ we have

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mu \left(\bigcap_{j=1}^n \alpha_j(x) \right) = \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mu \left(\bigcap_{j \in \mathbb{N}} \alpha_j(x) \right) = 0,$$

which shows (3.24). Now let us turn to the proof of (3.25).

Fix $\epsilon > 0$ and select $0 < \delta < 1$ such that $-\xi(\log \xi - \log k) < \epsilon$ for every $0 < \xi \leq \delta$. Observe that there is a finite set $J \subseteq I$ such that $\eta = \sum_{i \in J} \mu(B_i) \geq 1 - \delta$. Fix any $n \in \mathbb{N}$ and enumerate the elements of the partition $\alpha_1 \vee \cdots \vee \alpha_n = \{B_1^n, \dots, B_{l_n}^n\}$. Since each of $\alpha_1, \dots, \alpha_n$ contains at most k elements we have $l_n \leq k^n$. Put $C_j^n = B_j^n \setminus \{B_i : i \in J\}$ for $j = 1, \dots, n$ and write $\beta_n = \{C_j^n : 1 \leq j \leq l_n\} \cup \{B_i : i \in J\}$. Obviously $\beta_n \succeq \alpha_1 \vee \cdots \vee \alpha_n$, which gives

$$(3.26) \quad H_\mu \left(\bigvee_{j=1}^n \alpha_j \right) \leq H_\mu(\beta_n) = -\sum_{i \in J} \mu(B_i) \log \mu(B_i) - \sum_{j=1}^{l_n} \mu(C_j^n) \log \mu(C_j^n).$$

Using the convexity of $-x \log x$ on $[0, 1]$, and the definition of η we obtain

$$\begin{aligned} H_\mu \left(\bigvee_{j=1}^n \alpha_j \right) &\leq -\sum_{i \in J} \mu(B_i) \log \mu(B_i) - (1 - \eta) \log \frac{1 - \eta}{l_n} \text{ (using (3.26))} \\ &\leq -\sum_{i \in J} \mu(B_i) \log \mu(B_i) - (1 - \eta)(\log(1 - \eta) - n \log k) \\ &\leq -\sum_{i \in J} \mu(B_i) \log \mu(B_i) + n\epsilon. \end{aligned}$$

Dividing by n and letting $n \rightarrow \infty$, we see that $\lim_{n \rightarrow \infty} \frac{1}{n} H_\mu \left(\bigvee_{j=1}^n \alpha_j \right) \leq \epsilon$. Since $\epsilon > 0$ may be chosen arbitrarily small, the result follows. \square

4. CONTINUOUS BUNDLE RANDOM DYNAMICAL SYSTEMS

In this section we define and establish basic properties of a continuous bundle random dynamical system associated to an infinite countable discrete amenable group action, and give some known results for the special case of \mathbb{Z} from [8, 44, 53].

From now on, $(\Omega, \mathcal{F}, \mathbb{P}, G)$ will denote an MDS, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, that is, every subset of a null set of $(\Omega, \mathcal{F}, \mathbb{P})$ is measurable and has \mathbb{P} -measure 0.

Now let (X, \mathcal{B}) be a measurable space and $\mathcal{E} \in \mathcal{F} \times \mathcal{B}$. Set $\mathcal{E}_\omega = \{x \in X : (\omega, x) \in \mathcal{E}\}$ for each $\omega \in \Omega$. A *bundle random dynamical system* or *random dynamical system*

(RDS) for short *associated to* $(\Omega, \mathcal{F}, \mathbb{P}, G)$ is a family $\mathbf{F} = \{F_{g,\omega} : \mathcal{E}_\omega \rightarrow \mathcal{E}_{g\omega} | g \in G, \omega \in \Omega\}$ satisfying:

- (1) for each $\omega \in \Omega$, the transformation $F_{e_G,\omega}$ is the identity over \mathcal{E}_ω ,
- (2) for each $g \in G$, $(\omega, x) \mapsto F_{g,\omega}(x)$ is measurable and
- (3) for each $\omega \in \Omega$ and all $g_1, g_2 \in G$, $F_{g_2, g_1\omega} \circ F_{g_1,\omega} = F_{g_2 g_1, \omega}$ (and so $F_{g^{-1}, \omega} = (F_{g, g^{-1}\omega})^{-1}$ for each $g \in G$).

In this case, G has a natural measurable action on \mathcal{E} with $(\omega, x) \rightarrow (g\omega, F_{g,\omega}x)$ for each $g \in G$, called the corresponding *skew product transformation*.

Let the family $\mathbf{F} = \{F_{g,\omega} : \mathcal{E}_\omega \rightarrow \mathcal{E}_{g\omega} | g \in G, \omega \in \Omega\}$ be an RDS over $(\Omega, \mathcal{F}, \mathbb{P}, G)$, where X is a compact metric space with metric d and equipped with the Borel σ -algebra. If for \mathbb{P} -a.e. $\omega \in \Omega$, $\emptyset \neq \mathcal{E}_\omega \subseteq X$ is a compact subset and $F_{g,\omega}$ is a continuous map for each $g \in G$ (and so $F_{g,\omega} : \mathcal{E}_\omega \rightarrow \mathcal{E}_{g\omega}$ is a homeomorphism for \mathbb{P} -a.e. $\omega \in \Omega$ and each $g \in G$), then it is called a *continuous bundle RDS*.

By [12, Chapter III], the mapping $\omega \mapsto \mathcal{E}_\omega$ is measurable with respect to the Borel σ -algebra induced by the Hausdorff topology on the hyperspace 2^X of all non-empty compact subsets of X , and the distance function $d(x, \mathcal{E}_\omega)$ is measurable in $\omega \in \Omega$ for each $x \in X$.

Among interesting examples of continuous bundle RDSs are random sub-shifts.

In the case where $G = \mathbb{Z}$, these are treated in detail in [10, 41, 44]. We present a brief recall of some of their properties.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\vartheta : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\Omega, \mathcal{F}, \mathbb{P})$ an invertible measure-preserving transformation. Set $X = \{(x_i : i \in \mathbb{Z}) : x_i \in \mathbb{N} \cup \{+\infty\}, i \in \mathbb{Z}\}$ equipped with the metric

$$d((x_i : i \in \mathbb{Z}), (y_i : i \in \mathbb{Z})) = \sum_{i \in \mathbb{Z}} \frac{1}{2^{|i|}} |x_i^{-1} - y_i^{-1}|,$$

and let $F : X \rightarrow X$ be the translation $(x_i : i \in \mathbb{Z}) \mapsto (x_{i+1} : i \in \mathbb{Z})$. Then the integer group \mathbb{Z} acts on $\Omega \times X$ measurably with $(\omega, x) \mapsto (\vartheta^i \omega, F^i x)$ for each $i \in \mathbb{Z}$. Now let $\mathcal{E} \in \mathcal{F} \times \mathcal{B}_X$ be an invariant subset of $\Omega \times X$ (under the \mathbb{Z} -action) such that $\emptyset \neq \mathcal{E}_\omega \subseteq X$ is compact for \mathbb{P} -a.e. $\omega \in \Omega$. This defines a continuous bundle RDS where, for \mathbb{P} -a.e. $\omega \in \Omega$, $F_{i,\omega}$ is just the restriction of F^i over \mathcal{E}_ω for each $i \in \mathbb{Z}$.

A very special case is when the subset \mathcal{E} is given as follows. Let k be a random \mathbb{N} -valued random variable satisfying

$$0 < \int_{\Omega} \log k(\omega) d\mathbb{P}(\omega) < +\infty,$$

and, for \mathbb{P} -a.e. $\omega \in \Omega$, and let $M(\omega)$ be a random matrix $(m_{i,j}(\omega) : i = 1, \dots, k(\omega), j = 1, \dots, k(\vartheta\omega))$ with entries 0 and 1. Then the random variable k and the random matrix M generate a random sub-shift of finite type, where

$$\mathcal{E} = \{(\omega, (x_i : i \in \mathbb{Z})) : \omega \in \Omega, 1 \leq x_i \leq k(\vartheta^i \omega), m_{x_i, x_{i+1}}(\vartheta^i \omega) = 1, i \in \mathbb{Z}\}.$$

It is not hard to see that this is a continuous bundle RDS.

There are many other interesting examples of continuous bundle RDSs coming from smooth ergodic theory, see for example [49, 52], where one considers not only the action of the group \mathbb{Z} on a compact metric state space but also the semigroup \mathbb{Z}_+ on a Polish state space. (Recall that a *Polish space* is a complete separable metric space).

Let M be a C^∞ compact connected Riemannian manifold without boundary and $C^r(M, M)$, $r \in \mathbb{Z}_+ \cup \{+\infty\}$ the space of all C^r maps from M into itself endowed with

the usual C^r topology and the Borel σ -algebra. As above, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and $\vartheta : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\Omega, \mathcal{F}, \mathbb{P})$ is an invertible measure-preserving transformation. Now let $F : \Omega \rightarrow C^r(M, M)$ be a measurable map and define the family of the randomly composed maps $F_{n,\omega}$, $n \in \mathbb{Z}$ or \mathbb{Z}_+ , $\omega \in \Omega$ as follows:

$$F_{n,\omega} = \begin{cases} F(\vartheta^{n-1}\omega) \circ \dots \circ F(\vartheta\omega) \circ F(\omega), & \text{if } n > 0 \\ id, & \text{if } n = 0, \\ F(\vartheta^n\omega)^{-1} \circ \dots \circ F(\vartheta^{-2}\omega)^{-1} \circ F(\vartheta^{-1}\omega)^{-1}, & \text{if } n < 0 \end{cases}$$

here $F_{n,\omega}$, $n < 0$ is defined when $F(\omega) \in \text{Diff}^r(M)$ for \mathbb{P} -a.e. $\omega \in \Omega$. In the case of $r = 0$ we may replace M with a compact metric space.

Henceforth, we will fix the family $\mathbf{F} = \{F_{g,\omega} : \mathcal{E}_\omega \rightarrow \mathcal{E}_{g\omega} | g \in G, \omega \in \Omega\}$ to be a continuous bundle RDS over $(\Omega, \mathcal{F}, \mathbb{P}, G)$ with a compact metric space (X, d) as its state space.

As discussed in §3, one can introduce $\mathbf{C}_\mathcal{E}, \mathbf{P}_\mathcal{E}$ and other related notations. Moreover, for $S \subseteq \mathcal{E}$, if for \mathbb{P} -a.e. $\omega \in \Omega$ all fibers $S_\omega \subseteq \mathcal{E}_\omega$ are open or closed, then S is called an *open* or a *closed random set*. Denote by $\mathbf{C}_\mathcal{E}^o$ the set of all elements from $\mathbf{C}_\mathcal{E}$ consisting of subsets of open random sets. Similarly, we can introduce $\mathbf{C}_X, \mathbf{P}_X, \mathbf{C}_X^o$ and other related notations. Moreover, for $\xi \in \mathbf{C}_\Omega$ and $\mathcal{W} \in \mathbf{C}_X$, we introduce the notation

$$(\xi \times \mathcal{W})_\mathcal{E} = \{(C \times W) \cap \mathcal{E} : C \in \xi, W \in \mathcal{W}\} \in \mathbf{C}_\mathcal{E}.$$

In special cases, we will write $(\Omega \times \mathcal{W})_\mathcal{E} = (\{\Omega\} \times \mathcal{W})_\mathcal{E}$ and $(\xi \times X)_\mathcal{E} = (\xi \times \{X\})_\mathcal{E}$.

Denote by $\mathcal{P}_\mathbb{P}(\Omega \times X)$ the space of all probability measures on $\Omega \times X$ having the marginal \mathbb{P} on Ω . Every such a probability measure μ has the property that $\mu(A \times X) = \mathbb{P}(A)$ for each $A \in \mathcal{F}$. Put $\mathcal{P}_\mathbb{P}(\mathcal{E}) = \{\mu \in \mathcal{P}_\mathbb{P}(\Omega \times X) : \mu(\mathcal{E}) = 1\}$.

Recall that a topological space is σ -compact if it can be represented as a union of countably many compact subspaces.

For preparations, we need [26, Theorem 1].

Lemma 4.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and X a σ -compact Hausdorff space with $\pi : \Omega \times X \rightarrow \Omega$ the natural projection. If $A \in \mathcal{F} \times \mathcal{B}_X$ then there exists a measurable map $p : \Omega \rightarrow X$ such that $(\omega, p(\omega)) \in A$ for each $\omega \in \pi(A)$.*

Before proceeding, we also need the following result which is just a re-statement of [12, Theorem III.23]. We will use this often in the sequel.

Lemma 4.2. *Let (Γ, \mathcal{T}) be a measurable space and X a Polish space with $\pi : \Gamma \times X \rightarrow \Gamma$ the natural projection. Then $\pi(A) \in \mathcal{T}$ for each $A \in \mathcal{T} \times \mathcal{B}_X$.*

The following result is well known, but we were not able to find a suitable proof in the literature. We include here a proof for completeness.

Proposition 4.3. $\mathcal{P}_\mathbb{P}(\mathcal{E}) \neq \emptyset$.

Proof. Observe that $\mathcal{E} \in \mathcal{F} \times \mathcal{B}_X$ and $\mathcal{E}_\omega \neq \emptyset$ for \mathbb{P} -a.e. $\omega \in \Omega$. By Lemma 4.1 there exists a measurable map $p : \Omega \rightarrow X$ such that $(\omega, p\omega) \in \mathcal{E}$ for \mathbb{P} -a.e. $\omega \in \Omega$. Now we introduce μ over $\mathcal{F} \times \mathcal{B}_X$ as follows:

$$\mu(C) = \mathbb{P}(\pi(C \cap G_p)) \text{ for each } C \in \mathcal{F} \times \mathcal{B}_X,$$

where $\pi : \Omega \times X \rightarrow \Omega$ is the natural projection and $G_p = \{(\omega, p\omega) : \omega \in \Omega\} \in \mathcal{F} \times \mathcal{B}_X$ (as $p : \Omega \rightarrow X$ is measurable). By Lemma 4.2, μ is well defined. Moreover, it is not hard to check that μ is a probability measure over $\mathcal{F} \times \mathcal{B}_X$ and $\mu(\mathcal{E}) = 1, \mu(A \times X) = \mathbb{P}(A)$ for each $A \in \mathcal{F}$. That is, $\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E})$. Thus $\mathcal{P}_\mathbb{P}(\mathcal{E}) \neq \emptyset$. \square

Let $\mathcal{F}_{\mathcal{E}}$ be the σ -algebra of all sets of the form $(A \times X) \cap \mathcal{E}$, $A \in \mathcal{F}$. Note that each $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$ can be disintegrated as

$$d\mu(\omega, x) = d\mu_{\omega}(x)d\mathbb{P}(\omega),$$

where $\mu_{\omega}, \omega \in \Omega$ are regular conditional probability measures with respect to the σ -algebra $\mathcal{F}_{\mathcal{E}}$, that is, for \mathbb{P} -a.e. $\omega \in \Omega$, μ_{ω} is a Borel probability measure on \mathcal{E}_{ω} and, for any measurable subset $R \subseteq \mathcal{E}$,

$$(4.1) \quad \mu_{\omega}(R_{\omega}) = \mu(R|\mathcal{F}_{\mathcal{E}})(\omega)$$

where $R_{\omega} = \{x \in X : (\omega, x) \in R\}$. It follows that

$$(4.2) \quad \mu(R) = \int_{\Omega} \mu_{\omega}(R_{\omega})d\mathbb{P}(\omega).$$

For details see [24, Section 10.2].

Let $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$, $\alpha \in \mathbf{P}_{\mathcal{E}}$, $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}$. Then

$$(4.3) \quad H_{\mu}(\alpha|\mathcal{F}_{\mathcal{E}}) = - \int_{\Omega} \sum_{A \in \alpha} \mu(A|\mathcal{F}_{\mathcal{E}})(\omega) \log \mu(A|\mathcal{F}_{\mathcal{E}})(\omega) d\mathbb{P}(\omega)$$

$$(4.4) \quad = \int_{\Omega} H_{\mu_{\omega}}(\alpha_{\omega}) d\mathbb{P}(\omega) \text{ (using (4.1))},$$

here, $\alpha_{\omega} = \{A_{\omega} : A \in \alpha\}$ is a partition of \mathcal{E}_{ω} . In fact, by Lemma 3.15 we have

$$(4.5) \quad H_{\mu}(\mathcal{U}|\mathcal{F}_{\mathcal{E}}) = \int_{\Omega} H_{\mu_{\omega}}(\mathcal{U}_{\omega}) d\mathbb{P}(\omega).$$

Observe that, as in Remark 3.17, the assumption that Ω is a Lebesgue space in Lemma 3.15 is not essential, as in our setting we always have the disintegration $d\mu(\omega, x) = d\mu_{\omega}(x)d\mathbb{P}(\omega)$ of $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$ over $\mathcal{F}_{\mathcal{E}}$. Hence we can still obtain the equality (4.5). Note that for each $F \in \mathcal{F}_G$ and for any $\omega \in \Omega$, one has

$$(4.6) \quad (\mathcal{U}_F)_{\omega} = \bigvee_{g \in F} (g^{-1}\mathcal{U})_{\omega} = \bigvee_{g \in F} (F_{g, \omega})^{-1} \mathcal{U}_{g\omega} = \bigvee_{g \in F} F_{g^{-1}, g\omega} \mathcal{U}_{g\omega},$$

and so, in view of (4.5),

$$(4.7) \quad H_{\mu}(\mathcal{U}_F|\mathcal{F}_{\mathcal{E}}) = \int_{\Omega} H_{\mu_{\omega}} \left(\bigvee_{g \in F} F_{g^{-1}, g\omega} \mathcal{U}_{g\omega} \right) d\mathbb{P}(\omega).$$

Moreover, for any $\omega \in \Omega$, denote by $N(\mathcal{U}, \omega)$ the minimal cardinality of a sub-family of \mathcal{U}_{ω} covering \mathcal{E}_{ω} (i.e. the minimal cardinality of a sub-family of \mathcal{U} covering \mathcal{E}_{ω}), it is easy to check $H_{\mu_{\omega}}(\mathcal{U}_{\omega}) \leq \log N(\mathcal{U}, \omega)$.

Then we have:

Proposition 4.4. *Let $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}$. Then $N(\mathcal{U}, \omega)$ is measurable in $\omega \in \Omega$, and*

$$(4.8) \quad H_{\mu}(\mathcal{U}|\mathcal{F}_{\mathcal{E}}) \leq \int_{\Omega} \log N(\mathcal{U}, \omega) d\mathbb{P}(\omega).$$

Proof. We will call $\pi : \mathcal{E} \rightarrow \Omega$ to be the natural projection.

Let $n \in \mathbb{N}$. Then $N(\mathcal{U}, \omega) \leq n$ if and only if there exists U_1, \dots, U_n from \mathcal{U} such that $\mathcal{E}_{\omega} \subseteq \bigcup_{i=1}^n U_i$. Equivalently, $\omega \notin \pi(\mathcal{E} \setminus \bigcup_{i=1}^n U_i)$. Observe that for given U_1, \dots, U_n

the subset $\pi(\mathcal{E} \setminus \bigcup_{i=1}^n U_i)$ is measurable from Lemma 4.2. From this it is easy to see that $N(\mathcal{U}, \omega)$ is measurable in $\omega \in \Omega$, and hence we obtain the inequality (4.8). \square

Note that $\mathcal{F}_{\mathcal{E}} \subseteq (\mathcal{F} \times \mathcal{B}_X) \cap \mathcal{E}$ is a G -invariant sub- σ -algebra. It is not hard to check that, for $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$, μ is G -invariant if and only if $F_{g,\omega}\mu_{\omega} = \mu_{g\omega}$ for \mathbb{P} -a.e. $\omega \in \Omega$ and each $g \in G$, here $F_{g,\omega}\mu_{\omega}(\bullet)$ is given by $\mu_{\omega}(F_{g,\omega}^{-1}\bullet)$. (Observe that G acts naturally over \mathcal{E}). Denote by $\mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ the set of all G -invariant elements from $\mathcal{P}_{\mathbb{P}}(\mathcal{E})$. Just as the case of $(\Omega, \mathcal{F}, \mathcal{P})$ being trivial, in general $\mathcal{P}_{\mathbb{P}}(\mathcal{E}, G) \neq \emptyset$. A possible argument may be carried out as follows.

For each real-valued function f on \mathcal{E} which is measurable in $(\omega, x) \in \mathcal{E}$ and continuous in $x \in \mathcal{E}_{\omega}$ (for each fixed $\omega \in \Omega$), we set

$$\|f\|_1 = \int_{\Omega} \|f(\omega)\|_{\infty} d\mathbb{P}(\omega), \text{ where } \|f(\omega)\|_{\infty} = \sup_{x \in \mathcal{E}_{\omega}} |f(\omega, x)|.$$

Denote by $\mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ the space of all such functions with $\|f\|_1 < +\infty$, where we will identify two such functions f and g provided $\|f - g\|_1 = 0$. It is easy to check that $(\mathbf{L}_{\mathcal{E}}^1(\Omega, C(X)), \|\bullet\|_1)$ becomes a Banach space.

As we will see, the role of $\mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ in the set-up of a continuous bundle RDS is just that of $C(X)$ when we consider a topological G -action (X, G) (i.e. the group G acts on a compact metric space X).

We will introduce a weak star topology in $\mathcal{P}_{\mathbb{P}}(\mathcal{E})$ as follows. Let $\mu, \mu_n \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}), n \in \mathbb{N}$. Then the sequence $\{\mu_n : n \in \mathbb{N}\}$ converges to μ in $\mathcal{P}_{\mathbb{P}}(\mathcal{E})$ if and only if the sequence $\{\int_{\mathcal{E}} f d\mu_n : n \in \mathbb{N}\}$ converges to $\int_{\mathcal{E}} f d\mu$ for each $f \in \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ (obviously, $\int_{\mathcal{E}} f d\mu_n$ and $\int_{\mathcal{E}} f d\mu$ are well-defined from the above definitions).

It is known that $\mathcal{P}_{\mathbb{P}}(\mathcal{E})$ is a non-empty compact space in this weak star topology, see for example [44, Lemma 2.1 (i)]. Moreover, by [14, Theorem 5.6] one sees that $\mathcal{P}_{\mathbb{P}}(\mathcal{E})$ is also a metric space. In fact, a compatible metric over $\mathcal{P}_{\mathbb{P}}(\mathcal{E})$ was constructed in the proof of [14, Theorem 5.6].

As X is a compact metric space, $\mathcal{P}(X)$, the set of all Borel probability measures over X , equipped with the usual weak star topology, is also a compact metric space. Say $l : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathbb{R}$ to be a compatible metric over $\mathcal{P}(X)$. Then a compatible metric ρ over $\mathcal{P}_{\mathbb{P}}(\mathcal{E})$ can be given as follows. Let $\mu^i \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$ with $d\mu^i(\omega, x) = d\mu_{\omega}^i(x) d\mathbb{P}(\omega)$ the disintegration of μ^i over $\mathcal{F}_{\mathcal{E}}, i = 1, 2$. Observe that $\mu_{\omega}^1, \mu_{\omega}^2 \in \mathcal{P}(X)$ for \mathbb{P} -a.e. $\omega \in \Omega$. Then

$$\rho(\mu^1, \mu^2) = \int_{\Omega} l(\mu_{\omega}^1, \mu_{\omega}^2) d\mathbb{P}(\omega).$$

Recall that a non-empty subset of a topological space is *clopen* if it is not only a closed subset but also an open subset.

With the help of [2, Lemma 1.6.6], following the ideas from [44, Lemma 2.1] we have directly (for other variations of it see also the proof of [35, Lemma 3.4] or [48, Lemma 3.2]):

Proposition 4.5. *Let $\mathcal{P}_{\mathbb{P}}(\mathcal{E})$ be equipped with the above-defined weak star topology.*

(1) *Assume $\{\nu_n : n \in \mathbb{N}\} \subseteq \mathcal{P}_{\mathbb{P}}(\mathcal{E})$. Then the set of limit points of the sequence*

$$\{\mu_n \doteq \frac{1}{|F_n|} \sum_{g \in F_n} g\nu_n : n \in \mathbb{N}\}$$

is non-empty and is contained in $\mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$.

- (2) Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space. Let $\{\mu_n : n \in \mathbb{N}\}$ be a sequence in $\mathcal{P}_{\mathbb{P}}(\mathcal{E})$ converging to $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$ with $d\mu(\omega, x) = d\mu_\omega(x)d\mathbb{P}(\omega)$ the disintegration of μ over $\mathcal{F}_{\mathcal{E}}$. If $\alpha \in \mathbf{P}_{\mathcal{E}}$ satisfies that α_ω is a clopen partition of \mathcal{E}_ω (i.e. each element of α_ω is clopen) for \mathbb{P} -a.e. $\omega \in \Omega$, then

$$\limsup_{n \rightarrow \infty} H_{\mu_n}(\alpha|\mathcal{F}_{\mathcal{E}}) \leq H_{\mu}(\alpha|\mathcal{F}_{\mathcal{E}}).$$

Remark 4.6. We should observe that if $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ is ergodic then $(\Omega, \mathcal{F}, \mathbb{P}, G)$ is also ergodic. In other words, once $(\Omega, \mathcal{F}, \mathbb{P}, G)$ is not ergodic then each element from $\mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ is also not ergodic.

From now on, the topological space $\mathcal{P}_{\mathbb{P}}(\mathcal{E})$ (and its subspace $\mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$) is assumed to be equipped with the above weak star topology if there are no indications to the contrary.

Now let $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$. Observe that $\mathcal{F}_{\mathcal{E}} \subseteq (\mathcal{F} \times \mathcal{B}_X) \cap \mathcal{E}$ is a G -invariant sub- σ -algebra, so we can introduce the μ -fiber entropy of \mathbf{F} with respect to \mathcal{U} and $\mu, +$ -fiber entropy of \mathbf{F} with respect to \mathcal{U} , respectively, by

$$h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}) = h_{\mu}(G, \mathcal{U}|\mathcal{F}_{\mathcal{E}}) \text{ and } h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) = h_{\mu,+}(G, \mathcal{U}|\mathcal{F}_{\mathcal{E}}).$$

Thus $h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) \geq h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U})$. We define the μ -fiber entropy of \mathbf{F} as

$$h_{\mu}^{(r)}(\mathbf{F}) = \sup_{\alpha \in \mathbf{P}_{\mathcal{E}}} h_{\mu}^{(r)}(\mathbf{F}, \alpha).$$

From the definitions we have directly $h_{\mu}^{(r)}(\mathbf{F}) = h_{\mu}(G, \mathcal{E}|\mathcal{F}_{\mathcal{E}})$.

By Theorem 3.3 and Proposition 3.10 one has:

Proposition 4.7. Let $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space then $(\mathcal{E}, (\mathcal{F} \times \mathcal{B}_X) \cap \mathcal{E}, \mu)$ is also a Lebesgue space and so

- (1) $h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) = h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U})$ for each $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}$.
- (2) $h_{\mu}(G, \mathcal{E}) = h_{\mu}^{(r)}(\mathbf{F}) + h_{\mathbb{P}}(G, \Omega)$.

The following observation will be used below.

Lemma 4.8. Let $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$.

- (1) If $\alpha_1, \alpha_2 \in \mathbf{P}_{\mathcal{E}}$ satisfy $(\alpha_1)_{\omega} \succeq (\alpha_2)_{\omega}$ for \mathbb{P} -a.e. $\omega \in \Omega$ then $H_{\mu}(\alpha_1|\mathcal{F}_{\mathcal{E}}) \geq H_{\mu}(\alpha_2|\mathcal{F}_{\mathcal{E}})$ and $h_{\mu}^{(r)}(\mathbf{F}, \alpha_1) \geq h_{\mu}^{(r)}(\mathbf{F}, \alpha_2)$.
- (2) If $\alpha \in \mathbf{P}_{\mathcal{E}}$ and $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}$ satisfy $\alpha_{\omega} \succeq \mathcal{U}_{\omega}$ for \mathbb{P} -a.e. $\omega \in \Omega$ then there exists $\alpha' \in \mathbf{P}_{\mathcal{E}}$ such that $\alpha' \succeq \mathcal{U}$ and $\alpha'_{\omega} = \alpha_{\omega}$ for \mathbb{P} -a.e. $\omega \in \Omega$, and so $H_{\mu}(\alpha|\mathcal{F}_{\mathcal{E}}) = H_{\mu}(\alpha'|\mathcal{F}_{\mathcal{E}}) \geq H_{\mu}(\mathcal{U}|\mathcal{F}_{\mathcal{E}})$.
- (3) If $\mathcal{U}_1, \mathcal{U}_2 \in \mathbf{C}_{\mathcal{E}}$ satisfy $(\mathcal{U}_1)_{\omega} \succeq (\mathcal{U}_2)_{\omega}$ for \mathbb{P} -a.e. $\omega \in \Omega$ then $h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}_1) \geq h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}_2)$ and $h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}_1) \geq h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}_2)$.
- (4) If $\mathcal{U}_1, \mathcal{U}_2 \in \mathbf{C}_{\mathcal{E}}$ satisfy $(\mathcal{U}_1)_{\omega} = (\mathcal{U}_2)_{\omega}$ for \mathbb{P} -a.e. $\omega \in \Omega$ then $h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}_1) = h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}_2)$ and $h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}_1) = h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}_2)$.

Proof. (1) Say $d\mu(\omega, x) = d\mu_{\omega}(x)d\mathbb{P}(\omega)$ to be the disintegration of μ over $\mathcal{F}_{\mathcal{E}}$, then by (4.4) one has

$$(4.9) \quad H_{\mu}(\alpha_1|\mathcal{F}_{\mathcal{E}}) = \int_{\Omega} H_{\mu_{\omega}}((\alpha_1)_{\omega})d\mathbb{P}(\omega) \geq \int_{\Omega} H_{\mu_{\omega}}((\alpha_2)_{\omega})d\mathbb{P}(\omega) = H_{\mu}(\alpha_2|\mathcal{F}_{\mathcal{E}}).$$

Note that, if $F \in \mathcal{F}_G$, for $(\alpha_1)_F, (\alpha_2)_F \in \mathbf{P}_{\mathcal{E}}$, by assumptions one has $((\alpha_1)_F)_\omega \succeq ((\alpha_2)_F)_\omega$ for \mathbb{P} -a.e. $\omega \in \Omega$ and so as in (4.9) one has $H_\mu((\alpha_1)_F|\mathcal{F}_{\mathcal{E}}) \geq H_\mu((\alpha_2)_F|\mathcal{F}_{\mathcal{E}})$, which implies that $h_\mu^{(r)}(\mathbf{F}, \alpha_1) \geq h_\mu^{(r)}(\mathbf{F}, \alpha_2)$.

(2) Without loss of generality, we may assume that $\alpha_\omega \succeq \mathcal{U}_\omega$ for each $\omega \in \Omega$. We define $\pi : \Omega \times X \rightarrow \Omega$ to be the natural projection and $\alpha = \{A_1, \dots, A_n\}, \mathcal{U} = \{U_1, \dots, U_m\}, n, m \in \mathbb{N}$. Set

$$\Omega_{i,j} = \{\omega \in \Omega : \emptyset \neq (A_i)_\omega \subseteq (U_j)_\omega\}$$

for all $i = 1, \dots, n$ and $j = 1, \dots, m$. By assumption,

$$\Omega = \bigcap_{i=1}^n \bigcup_{j=1}^m \Omega_{i,j} = \bigcup_{j_1, \dots, j_n \in \{1, \dots, m\}} \bigcap_{i=1}^n \Omega_{i,j_i}.$$

In fact, $\Omega_{i,j} = \pi(A_i) \setminus \pi(A_i \setminus U_j)$ and so by Lemma 4.2 one has $\Omega_{i,j} \in \mathcal{F}$, thus there exists $\{\Omega_{j_1, \dots, j_n}^* : j_1, \dots, j_n \in \{1, \dots, m\}\} \in \mathbf{P}_\Omega$ such that $\Omega_{j_1, \dots, j_n}^* \subseteq \bigcap_{i=1}^n \Omega_{i,j_i}$ for all $j_1, \dots, j_n \in \{1, \dots, m\}$. Now set

$$\alpha' = \{(\Omega_{j_1, \dots, j_n}^* \times X) \cap A_i : i = 1, \dots, n, j_1, \dots, j_n \in \{1, \dots, m\}\}.$$

We claim that α' has the required property and hence using the definitions and (1) we obtain

$$H_\mu(\alpha|\mathcal{F}_{\mathcal{E}}) = H_\mu(\alpha'|\mathcal{F}_{\mathcal{E}}) \geq H_\mu(\mathcal{U}|\mathcal{F}_{\mathcal{E}}).$$

From the construction of α' , it is clear that $\alpha' \in \mathbf{P}_{\mathcal{E}}$ and $\alpha'_\omega = \alpha_\omega$ for each $\omega \in \Omega$. Moreover, if B is an atom of α' , say $B = (\Omega_{j_1, \dots, j_n}^* \times X) \cap A_s$, then $B \subseteq U_{j_s}$, as

$$\Omega_{j_1, \dots, j_n}^* \subseteq \bigcap_{i=1}^n \Omega_{i,j_i} \subseteq \Omega_{s,j_s}$$

and so

$$\begin{aligned} B &= \{(\omega, x) \in A_s : \omega \in \Omega_{j_1, \dots, j_n}^*\} \\ &\subseteq \{(\omega, x) \in A_s : \omega \in \Omega_{s,j_s}\} \\ &\subseteq \{(\omega, x) \in U_{j_s} : \omega \in \Omega_{s,j_s}\} \subseteq U_{j_s}, \end{aligned}$$

which proves that $\alpha' \succeq \mathcal{U}$.

(3) follows from (2) and (4) follows from (3). \square

Remark 4.9. By the construction of α' in the proof of Proposition 4.8 (2), we may take α' to be of the form $(\xi \times X)_\mathcal{E} \vee \alpha$ for some $\xi \in \mathbf{P}_\Omega$.

As a direct corollary, we have:

Proposition 4.10. Let $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$.

- (1) If $\mathcal{W} \in \mathbf{C}_\Omega$ then $h_{\mu,+}^{(r)}(\mathbf{F}, (\mathcal{W} \times X)_\mathcal{E}) = h_{\mu,+}^{(r)}(\mathbf{F}, (\{\Omega\} \times X)_\mathcal{E}) = 0$.
- (2) If $\xi \in \mathbf{P}_\Omega$ and $\mathcal{V} \in \mathbf{C}_X$ then

$$\inf_{\beta \in \mathbf{P}_X, \beta \succeq \mathcal{V}} h_\mu^{(r)}(\mathbf{F}, (\Omega \times \beta)_\mathcal{E}) \geq h_{\mu,+}^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V})_\mathcal{E}) = h_{\mu,+}^{(r)}(\mathbf{F}, (\xi \times \mathcal{V})_\mathcal{E}).$$

- (3) Assume that $\mathcal{U} \in \mathbf{C}_\mathcal{E}$ has the form $\mathcal{U} = \{(\Omega_i \times B_i)^c : i = 1, \dots, n\}, n \in \mathbb{N} \setminus \{1\}$ with $\Omega_i \in \mathcal{F}$ and $B_i \in \mathcal{B}_X$ for each $i = 1, \dots, n$. If $\mathbb{P}(\bigcap_{i=1}^n \Omega_i) = 0$ then $h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) = 0$.

Proof. We only need check (3). If $\mathcal{U} \in \mathbf{C}_\mathcal{E}$ has the form $\mathcal{U} = \{(\Omega_i \times B_i)^c : i = 1, \dots, n\}, n \in \mathbb{N} \setminus \{1\}$ with $\Omega_i \in \mathcal{F}$ and $B_i \in \mathcal{B}_X$ for each $i = 1, \dots, n$ and $\mathbb{P}(\bigcap_{i=1}^n \Omega_i) = 0$. Obviously, $\mathcal{W}^* \doteq (\{\Omega_1^c, \dots, \Omega_n^c\} \times X)_\mathcal{E} \in \mathbf{P}_\mathcal{E}$ satisfies $\mathcal{W}^* \succeq \mathcal{U}$ (in the sense of μ). Thus (using (1))

$$0 \leq h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) \leq h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{W}^*) = 0.$$

This completes the proof of (3). \square

The main result of this section is:

Theorem 4.11. *Let $\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E}, G)$. Then*

$$\begin{aligned} h_\mu^{(r)}(\mathbf{F}) &= \sup_{\mathcal{U} \in \mathbf{C}_\mathcal{E}} h_\mu^{(r)}(\mathbf{F}, \mathcal{U}) = \sup_{\mathcal{U} \in \mathbf{C}_\mathcal{E}} h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) \\ &= \sup_{\mathcal{U} \in \mathbf{C}_\mathcal{E}^\circ} h_\mu^{(r)}(\mathbf{F}, \mathcal{U}) = \sup_{\mathcal{U} \in \mathbf{C}_\mathcal{E}^\circ} h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) \\ &= \sup_{\mathcal{V} \in \mathbf{C}_X} h_\mu^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V})_\mathcal{E}) = \sup_{\mathcal{V} \in \mathbf{C}_X} h_{\mu,+}^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V})_\mathcal{E}) \\ &= \sup_{\mathcal{V} \in \mathbf{C}_X^\circ} h_\mu^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V})_\mathcal{E}) = \sup_{\mathcal{V} \in \mathbf{C}_X^\circ} h_{\mu,+}^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V})_\mathcal{E}). \end{aligned}$$

Proof. By the definitions, we only need to prove

$$(4.10) \quad h_\mu^{(r)}(\mathbf{F}) = \sup_{\alpha \in \mathbf{P}_X} h_\mu^{(r)}(\mathbf{F}, (\Omega \times \alpha)_\mathcal{E})$$

and, for each $\beta \in \mathbf{P}_X$,

$$(4.11) \quad h_\mu^{(r)}(\mathbf{F}, (\Omega \times \beta)_\mathcal{E}) \leq \sup_{\mathcal{V} \in \mathbf{C}_X^\circ} h_\mu^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V})_\mathcal{E}).$$

Observe that, for convenience, μ may be viewed as a probability measure over $(\Omega \times X, \mathcal{F} \times \mathcal{B}_X)$ and so $(\Omega \times X, \mathcal{F} \times \mathcal{B}_X, \mu, G)$ may be viewed as an MDS defined up to μ -null sets.

Let us first prove (4.10). Recall that $\mathcal{F} \times \mathcal{B}_X$ is the sub- σ -algebra generated by $A \times B, A \in \mathcal{F}$ and $B \in \mathcal{B}_X$, and note that $\mathcal{F} \times \{X\} \subseteq \mathcal{F} \times \mathcal{B}_X$ is a G -invariant sub- σ -algebra. By Proposition 3.1 (4),

$$(4.12) \quad h_\mu(G, \Omega \times X | \mathcal{F} \times \{X\}) = \sup_{\xi \in \mathbf{P}_\Omega} \sup_{\alpha \in \mathbf{P}_X} h_\mu(G, \xi \times \alpha | \mathcal{F} \times \{X\}).$$

Furthermore, it is easy to check that

$$(4.13) \quad h_\mu(G, \Omega \times X | \mathcal{F} \times \{X\}) = h_\mu^{(r)}(\mathbf{F}).$$

Now $d\mu(\omega, x) = d\mu_\omega(x)d\mathbb{P}(\omega)$ may also be viewed as the disintegration of μ over $\mathcal{F} \times \{X\}$, and hence, whenever $\xi \in \mathbf{P}_\Omega, \alpha \in \mathbf{P}_X$, one has (using reasoning similar to (4.7)),

$$\begin{aligned} h_\mu(G, \xi \times \alpha | \mathcal{F} \times \{X\}) &= \lim_{n \rightarrow \infty} \frac{1}{|F_n|} H_\mu((\xi \times \alpha)_{F_n} | \mathcal{F} \times \{X\}) \\ &= \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_\Omega H_{\mu_\omega} \left(\bigvee_{g \in F_n} F_{g^{-1}, g\omega} \alpha \right) d\mathbb{P}(\omega) \\ (4.14) \quad &= \lim_{n \rightarrow \infty} \frac{1}{|F_n|} H_\mu(((\Omega \times \alpha)_\mathcal{E})_{F_n} | \mathcal{F}_\mathcal{E}) = h_\mu^{(r)}(\mathbf{F}, (\Omega \times \alpha)_\mathcal{E}). \end{aligned}$$

Then (4.10) follows obviously from (4.12), (4.13) and (4.14).

Now we turn to the proof of (4.11).

Let $\beta \in \mathbf{P}_X, \epsilon > 0$ and say $\beta = \{B_1, \dots, B_n\}, n \in \mathbb{N}$. Observe that there exists $\delta > 0$ such that if $\xi = \{C_1, \dots, C_n\} \in \mathbf{P}_\mathcal{E}$ satisfies $\sum_{i=1}^n \mu((\Omega \times B_i) \cap \mathcal{E} \Delta C_i) < \delta$ then

$$H_\mu((\Omega \times \beta)_\mathcal{E} | \xi) + H_\mu(\xi | (\Omega \times \beta)_\mathcal{E}) < \epsilon.$$

Clearly for each $i = 1, \dots, n$ there exists compact $K_i \subseteq B_i$ such that $\mu(\Omega \times (B_i \setminus K_i)) < \frac{\delta}{n^2}$. Set $\mathcal{U} = \{K_i \cup U : i = 1, \dots, n\}$, where $U = X \setminus (K_1 \cup \dots \cup K_n)$. Then $\mathcal{U} \in \mathbf{C}_X^\circ$ and $\mu(\Omega \times U) < \frac{\delta}{n}$. Moreover, if $\gamma \in \mathbf{P}_\mathcal{E}$ satisfies $\gamma \succeq (\Omega \times \mathcal{U})_\mathcal{E}$ then there exists $\eta = \{A_1, \dots, A_n\} \in \mathbf{P}_\mathcal{E}$ such that $\gamma \succeq \eta$ and $A_i \subseteq \Omega \times (K_i \cup U)$ for each $i = 1, \dots, n$. Observe that by the selection of η one has $\Omega \times K_i \subseteq A_i$ (up to μ -null sets) and $K_i \subseteq B_i \subseteq K_i \cup U$ for each $i = 1, \dots, n$, and so

$$\sum_{i=1}^n \mu(A_i \Delta (\Omega \times B_i)) < n\mu(\Omega \times U) < \delta,$$

which implies

$$H_\mu((\Omega \times \beta)_\mathcal{E} | \gamma) \leq H_\mu((\Omega \times \beta)_\mathcal{E} | \eta) < \epsilon.$$

Now, for each $F \in \mathcal{F}_G$, if $\zeta \in \mathbf{P}_\mathcal{E}$ satisfies $\zeta \succeq ((\Omega \times \mathcal{U})_\mathcal{E})_F$ then $g\zeta \succeq (\Omega \times \mathcal{U})_\mathcal{E}$ and

$$H_\mu((\Omega \times \beta)_\mathcal{E} | g\zeta) < \epsilon$$

for each $g \in F$, thus

$$\begin{aligned} H_\mu(((\Omega \times \beta)_\mathcal{E})_F | \mathcal{F}_\mathcal{E}) &\leq H_\mu(\zeta | \mathcal{F}_\mathcal{E}) + H_\mu(((\Omega \times \beta)_\mathcal{E})_F | \zeta) \\ &\leq H_\mu(\zeta | \mathcal{F}_\mathcal{E}) + \sum_{g \in F} H_\mu((\Omega \times \beta)_\mathcal{E} | g\zeta) \\ &< H_\mu(\zeta | \mathcal{F}_\mathcal{E}) + |F|\epsilon, \end{aligned}$$

which implies

$$H_\mu(((\Omega \times \beta)_\mathcal{E})_F | \mathcal{F}_\mathcal{E}) \leq H_\mu(((\Omega \times \mathcal{U})_\mathcal{E})_F | \mathcal{F}_\mathcal{E}) + |F|\epsilon.$$

Last, for each $m \in \mathbb{N}$ substituting F by F_m , dividing both hands by $|F_m|$ and then letting m tend to infinity we obtain

$$h_\mu^{(r)}(\mathbf{F}, (\Omega \times \beta)_\mathcal{E}) \leq h_\mu^{(r)}(\mathbf{F}, (\Omega \times \mathcal{U})_\mathcal{E}) + \epsilon.$$

(4.11) follows easily from this. This completes the proof. \square

Part 2. Local Variational Principle for Fiber Topological Pressure

In this part we present and prove our main results. More precisely, given a continuous bundle random dynamical system associated to an infinite countable discrete amenable group action and a monotone sub-additive invariant family of random “continuous” functions, we introduce and discuss the local fiber topological pressure for a finite measurable cover, establish the associated variational principle which relates it to measure-theoretic entropy, under some necessary assumptions. We also discuss some special cases of the Theorem.

5. LOCAL FIBER TOPOLOGICAL PRESSURE

In this section, given a continuous bundle random dynamical system associated to an infinite countable discrete amenable group action and a monotone sub-additive invariant family of random “continuous” functions, we introduce the concept of the local fiber topological pressure for a finite measurable cover and discuss some basic properties. Our discussion follows the ideas of [38, 63, 74].

Let $f \in \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$. f is called *non-negative* if for \mathbb{P} -a.e. $\omega \in \Omega$, $f(\omega, x)$ is a non-negative function over \mathcal{E}_{ω} . Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\}$ be a family in $\mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$. We say that \mathbf{D} is

- (1) *non-negative* if each element from \mathbf{D} is non-negative;
- (2) *sub-additive* if for \mathbb{P} -a.e. $\omega \in \Omega$, $d_{E \cup Fg}(\omega, x) \leq d_E(\omega, x) + d_F(g(\omega, x))$ whenever $E, F \in \mathcal{F}_G$ and $g \in G$ satisfy $E \cap Fg = \emptyset$ and $x \in \mathcal{E}_{\omega}$;
- (3) *G-invariant* if for \mathbb{P} -a.e. $\omega \in \Omega$, $d_{Fg}(\omega, x) = d_F(g(\omega, x))$ whenever $F \in \mathcal{F}_G$, $g \in G$ and $x \in \mathcal{E}_{\omega}$;
- (4) *monotone* if for \mathbb{P} -a.e. $\omega \in \Omega$, $d_E(\omega, x) \leq d_F(\omega, x)$ whenever $E, F \in \mathcal{F}_G$ satisfy $E \subseteq F$ and $x \in \mathcal{E}_{\omega}$.

For example, for each $f \in \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$, it is easy to check that

$$\mathbf{D}^f \doteq \{d_F^f(\omega, x) \doteq \sum_{g \in F} f(g(\omega, x)) : F \in \mathcal{F}_G\}$$

is a sub-additive G -invariant family in $\mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$. Observe that in $\mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ not every sub-additive G -invariant family is in this form, in fact, if $f \in \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ then the following family is also sub-additive and G -invariant:

$$\{d_F(\omega, x) \doteq \sum_{g \in F} f(g(\omega, x)) + \sqrt{|F|} : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X)).$$

Similarly we can introduce these families in $L^1(\Omega, \mathcal{F}, \mathbb{P})$.

It is easy to check that:

Proposition 5.1. *Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a sub-additive G -invariant family and $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$. Then, for the function*

$$f : \mathcal{F}_G \rightarrow \mathbb{R}, F \mapsto \int_{\mathcal{E}} d_F(\omega, x) d\mu(\omega, x),$$

$f(Eg) = f(E)$ and $f(E \cup F) \leq f(E) + f(F)$ whenever $g \in G$ and $E, F \in \mathcal{F}_G$ satisfy $E \cap F = \emptyset$. Moreover, if \mathbf{D} is monotone then \mathbf{D} is non-negative, and so f is a monotone non-negative sub-additive G -invariant function.

A similar conclusion also holds if the family belongs to $L^1(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. We only need check that if \mathbf{D} is monotone, then \mathbf{D} is non-negative. In fact, this follows directly from the assumptions of sub-additivity and monotonicity.

Let $F \in \mathcal{F}_G$. Then for each $E \in \mathcal{F}_G$ satisfying $E \cap F = \emptyset$, by the assumptions of sub-additivity and monotonicity over \mathbf{D} we have: for \mathbb{P} -a.e. $\omega \in \Omega$,

$$d_E(\omega, x) \leq d_{E \cup F}(\omega, x) \leq d_E(\omega, x) + d_F(\omega, x),$$

and so $d_F(\omega, x) \geq 0$ for each $x \in \mathcal{E}_\omega$. This finishes our proof. \square

Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_\mathcal{E}^1(\Omega, C(X))$ be a sub-additive G -invariant family and $\mathcal{U} \in \mathbf{C}_\mathcal{E}$. For each $F \in \mathcal{F}_G$ and any $\omega \in \Omega$ we set

$$\begin{aligned} & P_\mathcal{E}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}) \\ &= \inf \left\{ \sum_{A(\omega) \in \alpha(\omega)} \sup_{x \in A(\omega)} e^{d_F(\omega, x)} : \alpha(\omega) \in \mathbf{P}_{\mathcal{E}_\omega}, \alpha(\omega) \succeq (\mathcal{U}_F)_\omega \right\} \\ (5.1) \quad &= \inf \left\{ \sum_{A(\omega) \in \alpha(\omega)} \sup_{x \in A(\omega)} e^{d_F(\omega, x)} : \alpha(\omega) \in \mathbf{P}_{\mathcal{E}_\omega}, \alpha(\omega) \succeq \bigvee_{g \in F} F_{g^{-1}, g\omega} \mathcal{U}_{g\omega} \right\}, \end{aligned}$$

where $\mathbf{P}_{\mathcal{E}_\omega}$ is introduced as in previous sections and (5.1) follows from (4.6).

In fact, it is easy to obtain an alternative expression for $P_\mathcal{E}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F})$ viz:

$$(5.2) \quad P_\mathcal{E}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}) = \inf \left\{ \sum_{A \in \alpha} \sup_{x \in A_\omega} e^{d_F(\omega, x)} : \alpha \succeq \mathcal{U}_F \right\}.$$

To see this, for $\alpha(\omega) \in \mathbf{P}_{\mathcal{E}_\omega}$ with $\alpha(\omega) \succeq (\mathcal{U}_F)_\omega$, define

$$\beta = \{\{\omega\} \times A : A \in \alpha(\omega)\} \cup \{U \setminus (\{\omega\} \times \mathcal{E}_\omega) : U \in \mathcal{P}(\mathcal{U}_F)\}.$$

Then it is clear that $\beta \in \mathbf{P}_\mathcal{E}$ (since the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete). Further, $\beta_\omega = \alpha(\omega)$, $\beta \succeq \mathcal{U}_F$ (as $\alpha(\omega) \succeq (\mathcal{U}_F)_\omega$ and $\mathcal{P}(\mathcal{U}_F) \succeq \mathcal{U}_F$).

Before proceeding, we need:

Lemma 5.2. *Let $\mathcal{U} \in \mathbf{C}_\mathcal{E}$ and $\omega \in \Omega$. Then $\mathbf{P}(\mathcal{U}_\omega) = \{\alpha_\omega : \alpha \in \mathbf{P}(\mathcal{U})\}$.*

Proof. Say $\mathcal{U} = \{U_1, \dots, U_n\}$, $n \in \mathbb{N}$. Then $\mathcal{U}_\omega = \{(U_1)_\omega, \dots, (U_n)_\omega\}$. Now for each $s = (s_1, \dots, s_n) \in \{0, 1\}^n$ we set $\mathcal{U}_s = \bigcap_{i=1}^n U_i(s_i)$, where $U_i(0) = U_i$ and $U_i(1) = U_i^c$. Then $\mathcal{P}(\mathcal{U}) = \{\mathcal{U}_s : s \in \{0, 1\}^n\}$. From this we obtain $\mathcal{P}(\mathcal{U})_\omega = \mathcal{P}(\mathcal{U}_\omega)$, as $(\mathcal{U}_s)_\omega = (\mathcal{U}_\omega)_s$ (where $(\mathcal{U}_\omega)_s$ is introduced similarly) for each $s \in \{0, 1\}^n$.

By the above discussions it is simple to prove $\mathbf{P}(\mathcal{U}_\omega) \supseteq \{\alpha_\omega : \alpha \in \mathbf{P}(\mathcal{U})\}$. Now we prove the other direction. That is, let $\beta(\omega) \in \mathbf{P}(\mathcal{U}_\omega)$, we find some $\beta' \in \mathbf{P}(\mathcal{U})$ such that $\beta'_\omega = \beta(\omega)$.

Suppose $\beta(\omega) = \{B_1, \dots, B_m\}$, $m \in \mathbb{N}$ with each $B_i \neq \emptyset$, $i = 1, \dots, m$, and set

$$\mathfrak{S} = \{s = (s_1, \dots, s_n) \in \{0, 1\}^n : (\mathcal{U}_s)_\omega \neq \emptyset\},$$

$$\mathfrak{S}_j = \{s = (s_1, \dots, s_n) \in \{0, 1\}^n : \emptyset \neq (\mathcal{U}_s)_\omega \subseteq B_j\}, j = 1, \dots, m.$$

As $\beta(\omega) \in \mathbf{P}(\mathcal{U}_\omega)$, obviously $\mathfrak{S}_i \cap \mathfrak{S}_j = \emptyset$ if $1 \leq i \neq j \leq m$ and

$$(5.3) \quad \bigcup_{j=1}^m \mathfrak{S}_j = \mathfrak{S} \text{ and } \bigcup_{s \in \mathfrak{S}_j} (\mathcal{U}_s)_\omega = B_j, j = 1, \dots, m.$$

Now put $\beta' = \{\mathcal{U}_s : s \in \{0, 1\}^n \setminus \mathfrak{S}\} \cup \{B'_1, \dots, B'_m\}$, where

$$B'_j = \bigcup_{s \in \mathfrak{S}_j} \mathcal{U}_s, j = 1, \dots, m.$$

Obviously, $\beta' \in \mathbf{P}_{\mathcal{E}}, \mathcal{P}(\mathcal{U}) \succeq \beta'$ and (using (5.3))

$$\beta'_\omega = \{(B'_1)_\omega, \dots, (B'_m)_\omega\} = \{B_1, \dots, B_m\} = \beta(\omega).$$

To finish the proof, we only need to check that $\beta' \succeq \mathcal{U}$. In fact, for $j = 1, \dots, m$, if $B_j \subseteq U_i$ for some $i = 1, \dots, n$, then, for each $s = (s_1, \dots, s_n) \in \mathfrak{S}_j$,

$$\emptyset \neq \mathcal{U}_s \cap U_i \subseteq U_i(s_i) \cap U_i$$

and so $s_i = 0$. This implies

$$\mathcal{U}_s \subseteq U_i(s_i) = U_i$$

and hence $B'_j \subseteq U_i$. That is, $\beta' \succeq \mathcal{U}$. \square

We have alternative formula for $P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F})$.

Proposition 5.3. *Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a sub-additive G -invariant family and $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}$, $F \in \mathcal{F}_G$, $\omega \in \Omega$. Then*

$$(5.4) \quad P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}) = \min \left\{ \sum_{A(\omega) \in \alpha(\omega)} \sup_{x \in A(\omega)} e^{d_F(\omega, x)} : \alpha(\omega) \in \mathbf{P}((\mathcal{U}_F)_\omega) \right\}$$

$$(5.5) \quad = \min \left\{ \sum_{A \in \alpha} \sup_{x \in A_\omega} e^{d_F(\omega, x)} : \alpha \in \mathbf{P}(\mathcal{U}_F) \right\}.$$

Proof. Note that (5.5) follows directly from Lemma 5.2 and (5.4). Thus we only need prove (5.4). We should point out that $d_F(\omega, x)$ is continuous in $x \in \mathcal{E}_\omega$ and $(\mathcal{U}_F)_\omega \in \mathbf{C}_{\mathcal{E}_\omega}$ (where $\mathbf{C}_{\mathcal{E}_\omega}$ is introduced as in previous sections). The proof will therefore be finished if we can prove that if f is a continuous function over \mathcal{E}_ω and $\mathcal{W} \in \mathbf{C}_{\mathcal{E}_\omega}$ then

$$(5.6) \quad \inf_{\gamma \in \mathbf{P}_{\mathcal{E}_\omega}, \gamma \succeq \mathcal{W}} \sum_{B \in \gamma} \sup_{x \in B} e^{f(x)} = \min_{\zeta \in \mathbf{P}(\mathcal{W})} \sum_{C \in \zeta} \sup_{x \in C} e^{f(x)},$$

where again $\mathbf{P}(\mathcal{W})$ is introduced as in previous discussions. However, this is just a basic fact which is not hard to obtain, and we omit its proof (for details see for example the proof of [38, Lemma 2.1]). This establishes (5.4) and so finishes our proof. \square

Thus:

Proposition 5.4. *Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a sub-additive G -invariant family and $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}$. Then*

- (1) *for each $F \in \mathcal{F}_G$, the function $P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F})$ is measurable in $\omega \in \Omega$.*
- (2) *$\{\log P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}) : F \in \mathcal{F}_G\}$ is a sub-additive G -invariant family in $L^1(\Omega, \mathcal{F}, \mathbb{P})$.*
- (3) *for the function $p : \mathcal{F}_G \rightarrow \mathbb{R}, F \mapsto \int_{\Omega} \log P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}) d\mathbb{P}(\omega)$, one has $p(Eg) = p(E)$ and $p(E \cup F) \leq p(E) + p(F)$ whenever $E, F \in \mathcal{F}_G$ and $g \in G$ satisfy $E \cap F = \emptyset$; moreover, if \mathbf{D} is monotone then p is a monotone non-negative G -invariant sub-additive function.*

Proof. (1) Let $F \in \mathcal{F}_G$. By (5.5), to prove the conclusion, we only need prove that $\sup_{x \in A_\omega} e^{d_F(\omega, x)}$ is measurable in $\omega \in \Omega$ for each $A \in (\mathcal{F} \times \mathcal{B}_X) \cap \mathcal{E}$. In fact, let $A \in (\mathcal{F} \times \mathcal{B}_X) \cap \mathcal{E}$ and say $\pi : \mathcal{E} \rightarrow \Omega$ to be the natural projection, then

$$\{\omega \in \Omega : \sup_{x \in A_\omega} e^{d_F(\omega, x)} > r\} = \pi(\{(\omega, x) \in A : e^{d_F(\omega, x)} > r\})$$

for each $r \in \mathbb{R}$, and so, by Lemma 4.2,

$$\{\omega \in \Omega : \sup_{x \in A_\omega} e^{d_F(\omega, x)} > r\}$$

is measurable, which implies that $\sup_{x \in A_\omega} e^{d_F(\omega, x)}$ is measurable in $\omega \in \Omega$.

(2) Let $E, F \in \mathcal{F}_G, g \in G$ satisfy $E \cap Fg = \emptyset$ and $\omega \in \Omega$. Then by (5.2) one has

$$\begin{aligned} e^{-\|d_E(\omega)\|_\infty} &\leq P_{\mathcal{E}}(\omega, \mathbf{D}, E, \mathcal{U}, \mathbf{F}) \\ &= \inf \left\{ \sum_{A \in \alpha} \sup_{x \in A_\omega} e^{d_E(\omega, x)} : \alpha \succeq \mathcal{U}_E \right\} \leq |\mathcal{U}_E| e^{\|d_E(\omega)\|_\infty}, \end{aligned}$$

which implies $\log P_{\mathcal{E}}(\omega, \mathbf{D}, E, \mathcal{U}, \mathbf{F}) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ (by the definition of $\mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$). Moreover, by the G -invariance of the family \mathbf{D} one has that, for \mathbb{P} -a.e. $\omega \in \Omega$,

$$\begin{aligned} P_{\mathcal{E}}(\omega, \mathbf{D}, Fg, \mathcal{U}, \mathbf{F}) &= \inf \left\{ \sum_{A \in \alpha} \sup_{x \in A_\omega} e^{d_{Fg}(\omega, x)} : \alpha \succeq \mathcal{U}_{Fg} \right\} \quad (\text{using (5.2)}) \\ &= \inf \left\{ \sum_{A \in \alpha} \sup_{x \in A_\omega} e^{d_F(g(\omega, x))} : g\alpha \succeq \mathcal{U}_F \right\} \\ (5.7) \quad &= \inf \left\{ \sum_{A \in \alpha} \sup_{x \in A_{g\omega}} e^{d_F(g\omega, x)} : \alpha \succeq \mathcal{U}_F \right\} = P_{\mathcal{E}}(g\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}), \end{aligned}$$

which implies the G -invariance of $\log P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F})$. Last, by the sub-additivity of the family \mathbf{D} and the G -invariance of $\log P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F})$, one has that, for \mathbb{P} -a.e. $\omega \in \Omega$,

$$\begin{aligned} &P_{\mathcal{E}}(\omega, \mathbf{D}, E \cup Fg, \mathcal{U}, \mathbf{F}) \\ &= \inf \left\{ \sum_{A \in \alpha} \sup_{x \in A_\omega} e^{d_{E \cup Fg}(\omega, x)} : \alpha \succeq \mathcal{U}_{E \cup Fg} \right\} \quad (\text{using (5.2)}) \\ &\leq \inf \left\{ \sum_{A \in \alpha, B \in \beta} \sup_{x \in A_\omega \cap B_\omega} e^{d_E(\omega, x) + d_F(g(\omega, x))} : \alpha \succeq \mathcal{U}_E, \beta \succeq \mathcal{U}_{Fg} \right\} \\ &\leq \inf \left\{ \sum_{A \in \alpha, B \in \beta} \sup_{x \in A_\omega} e^{d_E(\omega, x)} \sup_{x \in B_\omega} e^{d_F(g(\omega, x))} : \alpha \succeq \mathcal{U}_E, \beta \succeq \mathcal{U}_{Fg} \right\} \\ &= \inf \left\{ \sum_{A \in \alpha} \sup_{x \in A_\omega} e^{d_E(\omega, x)} : \alpha \succeq \mathcal{U}_E \right\} \inf \left\{ \sum_{B \in \beta} \sup_{x \in B_\omega} e^{d_F(g(\omega, x))} : \beta \succeq \mathcal{U}_{Fg} \right\} \\ &= P_{\mathcal{E}}(\omega, \mathbf{D}, E, \mathcal{U}, \mathbf{F}) P_{\mathcal{E}}(g\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}) \quad (\text{using (5.2) and (5.7)}), \end{aligned}$$

which implies the sub-additivity.

(3) follows directly from Proposition 5.1 and (2). \square

Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_\mathcal{E}^1(\Omega, C(X))$ be a monotone sub-additive G -invariant family and $\mathcal{U} \in \mathbf{C}_\mathcal{E}$. Then by Proposition 2.2 and Proposition 5.4 we define the *fiber topological \mathbf{D} -pressure of \mathbf{F} with respect to \mathcal{U}* and the *fiber topological \mathbf{D} -pressure of \mathbf{F}* , respectively, by

$$P_\mathcal{E}(\mathbf{D}, \mathcal{U}, \mathbf{F}) = \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_\Omega \log P_\mathcal{E}(\omega, \mathbf{D}, F_n, \mathcal{U}, \mathbf{F}) d\mathbb{P}(\omega)$$

and

$$P_\mathcal{E}(\mathbf{D}, \mathbf{F}) = \sup_{\mathcal{V} \in \mathbf{C}_X^\circ} P_\mathcal{E}(\mathbf{D}, (\Omega \times \mathcal{V})_\mathcal{E}, \mathbf{F}).$$

Obviously, \mathbf{D}^0 is a monotone sub-additive G -invariant family. It is direct to check

$$P_\mathcal{E}(\omega, \mathbf{D}^0, F, \mathcal{U}, \mathbf{F}) = N(\mathcal{U}_F, \omega)$$

whenever $\omega \in \Omega$ and $F \in \mathcal{F}_G$, and so one has

$$(5.8) \quad P_\mathcal{E}(\mathbf{D}^0, \mathcal{U}, \mathbf{F}) = \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_\Omega \log N(\mathcal{U}_{F_n}, \omega) d\mathbb{P}(\omega),$$

which is called the *fiber topological entropy of \mathbf{F} with respect to \mathcal{U}* (also denoted by $h_{\text{top}}^{(r)}(\mathbf{F}, \mathcal{U})$). Moreover, $P_\mathcal{E}(\mathbf{D}^0, \mathbf{F})$ is called the *fiber topological entropy of \mathbf{F}* (also denoted by $h_{\text{top}}^{(r)}(\mathbf{F})$). Remark that by Proposition 2.2 the values of all these invariants are independent of the selection of the Følner sequence $\{F_n : n \in \mathbb{N}\}$.

Before proceeding, recall [74, Lemma 2.1] (the only difference is that each of p_1, \dots, p_k may take value of 0 here).

Lemma 5.5. *Let $a_1, p_1, \dots, a_k, p_k \in \mathbb{R}$ with $p_1, \dots, p_k \geq 0$ and $\sum_{i=1}^k p_i = p$. Then*

$$\sum_{i=1}^k p_i(a_i - \log p_i) \leq p \log \left(\sum_{i=1}^k e^{a_i} \right) - p \log p.$$

The identity holds if and only if $p_i = \frac{pe^{a_i}}{\sum_{j=1}^k e^{a_j}}$ for each $i = 1, \dots, k$. In particular,

$$\sum_{i=1}^k -p_i \log p_i \leq p \log k - p \log p.$$

It is not too hard to see:

Proposition 5.6. *Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_\mathcal{E}^1(\Omega, C(X))$ be a sub-additive G -invariant family, $\mathcal{U} \in \mathbf{C}_\mathcal{E}$ and $\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E}, G)$ with $d\mu(\omega, x) = d\mu_\omega(x) d\mathbb{P}(\omega)$ the disintegration of μ over $\mathcal{F}_\mathcal{E}$.*

- (1) *Let $\omega \in \Omega$. If ν_ω is a Borel probability measure over \mathcal{E}_ω , then, for each $F \in \mathcal{F}_G$,*

$$H_{\nu_\omega}((\mathcal{U}_F)_\omega) + \int_{\mathcal{E}_\omega} d_F(\omega, x) d\nu_\omega(x) \leq \log P_\mathcal{E}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}).$$

- (2) *If \mathbf{D} is monotone then $P_\mathcal{E}(\mathbf{D}, \mathcal{U}, \mathbf{F}) \geq h_\mu^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D})$, where*

$$\mu(\mathbf{D}) = \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_\mathcal{E} d_{F_n}(\omega, x) d\mu(\omega, x) \geq 0,$$

observe that by Proposition 2.2 and Proposition 5.4 the limit must exist (and its value is independent of the selection of the Følner sequence $\{F_n : n \in \mathbb{N}\}$), and so $P_{\mathcal{E}}(\mathbf{D}, \mathbf{F}) \geq h_{\mu}^{(r)}(\mathbf{F}) + \mu(\mathbf{D})$ (using Theorem 4.11). In particular,

$$h_{top}^{(r)}(\mathbf{F}, \mathcal{U}) \geq h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}) \text{ and } h_{top}^{(r)}(\mathbf{F}) \geq h_{\mu}^{(r)}(\mathbf{F}).$$

Proof. (1) In fact, using Lemma 5.5 one has

$$\begin{aligned} & \log P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}) \\ &= \inf \log \left\{ \sum_{A(\omega) \in \alpha(\omega)} \sup_{x \in A(\omega)} e^{d_F(\omega, x)} : \alpha(\omega) \in \mathbf{P}_{\mathcal{E}_{\omega}}, \alpha(\omega) \succeq (\mathcal{U}_F)_{\omega} \right\} \\ &\geq \inf_{\alpha(\omega) \in \mathbf{P}_{\mathcal{E}_{\omega}}, \alpha(\omega) \succeq (\mathcal{U}_F)_{\omega}} \sum_{A(\omega) \in \alpha(\omega)} \nu_{\omega}(A(\omega)) \left(\sup_{x \in A(\omega)} d_F(\omega, x) - \log \nu_{\omega}(A(\omega)) \right) \\ &\geq \inf_{\alpha(\omega) \in \mathbf{P}_{\mathcal{E}_{\omega}}, \alpha(\omega) \succeq (\mathcal{U}_F)_{\omega}} \left\{ \int_{\mathcal{E}_{\omega}} d_F(\omega, x) d\nu_{\omega}(x) + H_{\nu_{\omega}}(\alpha(\omega)) \right\} \\ &= H_{\nu_{\omega}}((\mathcal{U}_F)_{\omega}) + \int_{\mathcal{E}_{\omega}} d_F(\omega, x) d\nu_{\omega}(x). \end{aligned}$$

(2) follows from (1) and the definitions. \square

Observe that if $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ is a monotone sub-additive G -invariant family then it is not hard to check that the family

$$\left\{ \sup_{x \in \mathcal{E}_{\omega}} d_F(\omega, x) = \|d_F(\omega)\|_{\infty} : F \in \mathcal{F} \right\} \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$$

is also monotone sub-additive and G -invariant. Hence we may define

$$\sup_{\mathbb{P}}(\mathbf{D}) = \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\Omega} \sup_{x \in \mathcal{E}_{\omega}} d_F(\omega, x) d\mathbb{P}(\omega) \geq \mu(\mathbf{D}).$$

Remark that by Proposition 2.2 and Proposition 5.4, the limit is well-defined and its value is independent of the selection of the Følner sequence $\{F_n : n \in \mathbb{N}\}$.

From the definition, it is easy to see:

Lemma 5.7. *Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a monotone sub-additive G -invariant family and $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$. Then*

$$\sup_{\mathbb{P}}(\mathbf{D}) \geq \limsup_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\mathcal{E}} d_{F_n}(\omega, x) d\mu(\omega, x).$$

As in Lemma 4.8 and Proposition 4.10, one has:

Proposition 5.8. *Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a sub-additive G -invariant family and $\mathcal{U}, \mathcal{U}_1, \mathcal{U}_2 \in \mathbf{C}_{\mathcal{E}}$.*

(1) *Let $\omega \in \Omega$ and $F \in \mathcal{F}_G$. Then*

$$\sup_{x \in \mathcal{E}_{\omega}} e^{d_F(\omega, x)} \leq P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}) \leq N(\mathcal{U}_F, \omega) \sup_{x \in \mathcal{E}_{\omega}} e^{d_F(\omega, x)}.$$

(2) *If $(\mathcal{U}_1)_{\omega} \succeq (\mathcal{U}_2)_{\omega}$ for \mathbb{P} -a.e. $\omega \in \Omega$, then*

$$\log P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}_1, \mathbf{F}) \geq \log P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}_2, \mathbf{F})$$

for \mathbb{P} -a.e. $\omega \in \Omega$ and each $F \in \mathcal{F}_G$.

(3) If $(\mathcal{U}_1)_\omega = (\mathcal{U}_2)_\omega$ for \mathbb{P} -a.e. $\omega \in \Omega$, then

$$\log P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}_1, \mathbf{F}) = \log P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}_2, \mathbf{F})$$

for \mathbb{P} -a.e. $\omega \in \Omega$ and each $F \in \mathcal{F}_G$.

(4) If \mathbf{D} is monotone then, for \mathbb{P} -a.e. $\omega \in \Omega$ and each $F \in \mathcal{F}_G$,

$$e^{\|d_F(\omega)\|_\infty} \leq P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}) \leq N(\mathcal{U}_F, \omega) e^{\|d_F(\omega)\|_\infty},$$

and hence

$$\sup_{\mathbb{P}}(\mathbf{D}) \leq P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) \leq h_{top}^{(r)}(\mathbf{F}, \mathcal{U}) + \sup_{\mathbb{P}}(\mathbf{D}).$$

(5) Assume that \mathcal{U} is in the form of $\mathcal{U} = \{(\Omega_i \times B_i)^c : i = 1, \dots, n\}$, $n \in \mathbb{N} \setminus \{1\}$ with $\Omega_i \in \mathcal{F}$ and $B_i \in \mathcal{B}_X$ for each $i = 1, \dots, n$. If $\mathbb{P}(\bigcap_{i=1}^n \Omega_i) = 0$ then

$h_{top}^{(r)}(\mathbf{F}, \mathcal{U}) = 0$, and so if, additionally, \mathbf{D} is monotone, then

$$P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) = \sup_{\mathbb{P}}(\mathbf{D}).$$

As a direct corollary, we have:

Corollary 5.9. Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a monotone sub-additive G -invariant family. Then

$$P_{\mathcal{E}}(\mathbf{D}, \mathbf{F}) = \sup_{\xi \in \mathbf{P}_{\Omega}, \mathcal{V} \in \mathbf{C}_X^o} P_{\mathcal{E}}(\mathbf{D}, (\xi \times \mathcal{V})_{\mathcal{E}}, \mathbf{F}).$$

Question 5.10. Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a monotone sub-additive G -invariant family. Do we have

$$P_{\mathcal{E}}(\mathbf{D}, \mathbf{F}) = \sup_{\mathcal{U} \in \mathbf{C}_{\mathcal{E}}^o} P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F})?$$

Observe that if, additionally, Ω is a compact metric space with $\mathcal{F} = \mathcal{B}_{\Omega}$ and $\mathcal{U} \in \mathbf{C}_{\Omega \times X}^o$, it is not hard to find $\mathcal{W} \in \mathbf{C}_{\Omega}^o$ and $\mathcal{V} \in \mathbf{C}_X^o$ with $\mathcal{W} \times \mathcal{V} \succeq \mathcal{U}$, and hence $\xi \times \mathcal{V} \succeq \mathcal{U}$ for some $\xi \in \mathbf{P}_{\Omega}$, thus, using Corollary 5.9 one has

$$P_{\mathcal{E}}(\mathbf{D}, \mathbf{F}) = \sup_{\mathcal{U} \in \mathbf{C}_{\mathcal{E}}^{t,o}} P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}).$$

Here, we denote by $\mathbf{C}_{\mathcal{E}}^{t,o}$ the set of all $\mathcal{U} \in \mathbf{C}_{\Omega \times X}^o$. (It is clear that $\mathbf{C}_{\mathcal{E}}^{t,o} \subseteq \mathbf{C}_{\mathcal{E}}^o$).

6. FACTOR EXCELLENT AND GOOD COVERS

In this section we introduce and discuss the concept of factor excellent and good covers which are one of two necessary assumptions in our main result Theorem 7.1. As shown by Theorem 6.9 and Theorem 6.10, many interesting covers are included in this special class of finite measurable covers.

Recall that a topological space is *zero-dimensional* if it has a topological base consisting of clopen subsets. Observe that, for a zero-dimensional compact metric space, the set of all clopen subsets is countable.

Let $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}$. Say $\mathcal{U} = \{U_1, \dots, U_N\}$, $N \in \mathbb{N}$. Set

$$\mathbf{P}_{\mathcal{U}} = \{\{A_1, \dots, A_N\} \in \mathbf{P}_{\mathcal{E}} : A_i \subseteq U_i, i = 1, \dots, N\}.$$

Before proceeding, we shall state a well-known fact.

Lemma 6.1. *Let Z be a zero-dimensional compact metric space and $\mathcal{W} \in \mathbf{C}_Z^o$. Set*

$$\mathbf{P}_c(\mathcal{W}) = \{\beta \in \mathbf{P}_{\mathcal{W}} : \beta \text{ is clopen}\},$$

where $\mathbf{P}_{\mathcal{W}}$ is introduced similarly. Then $\mathbf{P}_c(\mathcal{W})$ is a countable family and, for each $\gamma \in \mathbf{P}_{\mathcal{W}}$, if (Z, \mathcal{B}_Z, η) is a probability space then

$$\inf_{\beta \in \mathbf{P}_c(\mathcal{W})} [H_{\eta}(\gamma|\beta) + H_{\eta}(\beta|\gamma)] = 0.$$

We remark that the basic Lemma 6.1 serves as an important bridge in the establishment of the local entropy theory of \mathbb{Z} -actions and more generally for a countable discrete amenable group action (that is, first we obtain results for a zero-dimensional dynamical system, then by virtue of Lemma 6.1, we may generalize them to the general case).

Inspired by this, we introduce the following concepts which serve as one of the two essential assumptions in our main results presented later.

Let $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}$. \mathcal{U} is called *excellent* (*good*, respectively) if there exists a sequence $\{\alpha_n : n \in \mathbb{N}\} \subseteq \mathbf{P}_{\mathcal{U}}$ satisfying properties (1) and (2) (properties (1) and (3), respectively), where

- (1) for each $n \in \mathbb{N}$, $(\alpha_n)_{\omega}$ is a clopen partition of \mathcal{E}_{ω} for \mathbb{P} -a.e. $\omega \in \Omega$;
- (2) for each $\beta \in \mathbf{P}_{\mathcal{U}}$, if $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$ then

$$\inf_{n \in \mathbb{N}} [H_{\mu}(\beta|\alpha_n \vee \mathcal{F}_{\mathcal{E}}) + H_{\mu}(\alpha_n|\beta \vee \mathcal{F}_{\mathcal{E}})] = 0,$$

in fact, if say $d\mu(\omega, x) = d\mu_{\omega}(x)d\mathbb{P}(\omega)$ to be the disintegration of μ over $\mathcal{F}_{\mathcal{E}}$, then using (3.1) and (4.4) it is equivalent to

$$\inf_{n \in \mathbb{N}} \int_{\Omega} [H_{\mu_{\omega}}(\beta_{\omega}|(\alpha_n)_{\omega}) + H_{\mu_{\omega}}((\alpha_n)_{\omega}|\beta_{\omega})] d\mathbb{P}(\omega) = 0.$$

- (3) for each $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$, $h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) = \inf_{n \in \mathbb{N}} h_{\mu}^{(r)}(\mathbf{F}, \alpha_n)$, equivalently, for each $\beta \in \mathbf{P}_{\mathcal{U}}$, $h_{\mu}^{(r)}(\mathbf{F}, \beta) \geq \inf_{n \in \mathbb{N}} h_{\mu}^{(r)}(\mathbf{F}, \alpha_n)$.

By Proposition 3.1 (4) property of excellent is stronger than property of good.

It is easy to check:

Lemma 6.2. *Let $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}^o$. If there exists $\mathcal{U}' \in \mathbf{C}_{\mathcal{E}}^o$ such that $\mathcal{U}' \succeq \mathcal{U}$, \mathcal{U}' is good and $h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}') = h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U})$ for each $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$. Then \mathcal{U} is also good.*

We also have:

Lemma 6.3. *Let $\mathcal{U}_1, \mathcal{U}_2 \in \mathbf{C}_{\mathcal{E}}^o$ and $W \in \mathcal{F}$. If both \mathcal{U}_1 and \mathcal{U}_2 are excellent then $\mathcal{U}_1 \vee \mathcal{U}_2, \mathcal{U}_1 \cap (W \times X) \cup \mathcal{U}_2 \cap (W^c \times X) \in \mathbf{C}_{\mathcal{E}}^o$ and both of them are excellent.*

Proof. Obviously, $\mathcal{U}_1 \vee \mathcal{U}_2, \mathcal{U}_1 \cap (W \times X) \cup \mathcal{U}_2 \cap (W^c \times X) \in \mathbf{C}_{\mathcal{E}}^o$.

By assumption, for each $i = 1, 2$, there exists $\{\alpha_n^i : n \in \mathbb{N}\} \subseteq \mathbf{P}_{\mathcal{U}_i}$ satisfying

- (1) for each $n \in \mathbb{N}$, $(\alpha_n^i)_{\omega}$ is a clopen partition of \mathcal{E}_{ω} for \mathbb{P} -a.e. $\omega \in \Omega$ and
- (2) for each $\beta^i \in \mathbf{P}_{\mathcal{U}_i}$ and any $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$,

$$\inf_{n \in \mathbb{N}} [H_{\mu}(\beta^i|\alpha_n^i \vee \mathcal{F}_{\mathcal{E}}) + H_{\mu}(\alpha_n^i|\beta^i \vee \mathcal{F}_{\mathcal{E}})] = 0.$$

First we consider $\mathcal{U}_1 \vee \mathcal{U}_2$. For each $n_1, n_2 \in \mathbb{N}$ set $\alpha_{n_1, n_2} = \alpha_{n_1}^1 \vee \alpha_{n_2}^2$, it is clear that $\alpha_{n_1, n_2} \in \mathbf{C}_{\mathcal{E}}^o$ and $(\alpha_{n_1, n_2})_{\omega}$ is a clopen partition of \mathcal{E}_{ω} for \mathbb{P} -a.e. $\omega \in \Omega$. Now let $\beta \in \mathbf{P}_{\mathcal{U}_1 \vee \mathcal{U}_2}$. Suppose that $\beta = \{B_{U_1, U_2} \subseteq U_1 \cap U_2 : U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2\}$. Set

$$\beta^1 = \left\{ \bigcup_{U_2 \in \mathcal{U}_2} B_{U_1, U_2} : U_1 \in \mathcal{U}_1 \right\} \text{ and } \beta^2 = \left\{ \bigcup_{U_1 \in \mathcal{U}_1} B_{U_1, U_2} : U_2 \in \mathcal{U}_2 \right\}.$$

Then $\beta^i \in \mathbf{P}_{\mathcal{U}_i}$, $i = 1, 2$ and $\beta = \beta^1 \vee \beta^2$. Let $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$. So

$$\begin{aligned} & \inf_{n_1, n_2 \in \mathbb{N}} [H_{\mu}(\beta | \alpha_{n_1, n_2} \vee \mathcal{F}_{\mathcal{E}}) + H_{\mu}(\alpha_{n_1, n_2} | \beta \vee \mathcal{F}_{\mathcal{E}})] \\ &= \inf_{n_1, n_2 \in \mathbb{N}} [H_{\mu}(\beta^1 \vee \beta^2 | \alpha_{n_1}^1 \vee \alpha_{n_2}^2 \vee \mathcal{F}_{\mathcal{E}}) + H_{\mu}(\alpha_{n_1}^1 \vee \alpha_{n_2}^2 | \beta^1 \vee \beta^2 \vee \mathcal{F}_{\mathcal{E}})] \\ &\leq \inf_{n_1, n_2 \in \mathbb{N}} [H_{\mu}(\beta^1 | \alpha_{n_1}^1 \vee \mathcal{F}_{\mathcal{E}}) + H_{\mu}(\beta^2 | \alpha_{n_2}^2 \vee \mathcal{F}_{\mathcal{E}}) \\ &\quad + H_{\mu}(\alpha_{n_1}^1 | \beta^1 \vee \mathcal{F}_{\mathcal{E}}) + H_{\mu}(\alpha_{n_2}^2 | \beta^2 \vee \mathcal{F}_{\mathcal{E}})], \end{aligned}$$

by the construction of $\{\alpha_n^i : n \in \mathbb{N}\} \subseteq \mathbf{P}_{\mathcal{U}_i}$, $i = 1, 2$ one has

$$\inf_{n_1, n_2 \in \mathbb{N}} [H_{\mu}(\beta | \alpha_{n_1, n_2} \vee \mathcal{F}_{\mathcal{E}}) + H_{\mu}(\alpha_{n_1, n_2} | \beta \vee \mathcal{F}_{\mathcal{E}})] = 0.$$

That is, $\mathcal{U}_1 \vee \mathcal{U}_2$ is excellent.

Now let us consider $\mathcal{U} \doteq \mathcal{U}_1 \cap (W \times X) \cup \mathcal{U}_2 \cap (W^c \times X)$.

For each $n_1, n_2 \in \mathbb{N}$ set $\alpha_{n_1, n_2} = \alpha_{n_1}^1 \cap (W \times X) \cup \alpha_{n_2}^2 \cap (W^c \times X)$, obviously $\alpha_{n_1, n_2} \in \mathbf{C}_{\mathcal{E}}^o$ and $(\alpha_{n_1, n_2})_{\omega}$ is a clopen partition of \mathcal{E}_{ω} for \mathbb{P} -a.e. $\omega \in \Omega$. Let $\beta \in \mathbf{P}_{\mathcal{U}}$. It is easy to choose $\beta^i \in \mathbf{P}_{\mathcal{U}_i}$, $i = 1, 2$ such that $\beta = \beta^1 \cap (W \times X) \cup \beta^2 \cap (W^c \times X)$. So if $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$, say $d\mu(\omega, x) = d\mu_{\omega}(x)d\mathbb{P}(\omega)$ to be the disintegration of μ over $\mathcal{F}_{\mathcal{E}}$, then by the construction of $\{\alpha_n^i : n \in \mathbb{N}\} \subseteq \mathbf{P}_{\mathcal{U}_i}$, $i = 1, 2$ and (3.1), (4.4) one has

$$(6.1) \quad \inf_{n \in \mathbb{N}} \int_{\Omega} [H_{\mu_{\omega}}((\beta^i)_{\omega} | (\alpha_n^i)_{\omega}) + H_{\mu_{\omega}}((\alpha_n^i)_{\omega} | (\beta^i)_{\omega})] d\mathbb{P}(\omega) = 0, i = 1, 2$$

and (by the construction of $\beta^1, \beta^2, \alpha_{n_1, n_2}, n_1, n_2 \in \mathbb{N}$)

$$\begin{aligned} & \inf_{n_1, n_2 \in \mathbb{N}} [H_{\mu}(\beta | \alpha_{n_1, n_2} \vee \mathcal{F}_{\mathcal{E}}) + H_{\mu}(\alpha_{n_1, n_2} | \beta \vee \mathcal{F}_{\mathcal{E}})] \\ &= \inf_{n_1, n_2 \in \mathbb{N}} \int_{\Omega} [H_{\mu_{\omega}}(\beta_{\omega} | (\alpha_{n_1, n_2})_{\omega}) + H_{\mu_{\omega}}((\alpha_{n_1, n_2})_{\omega} | \beta_{\omega})] d\mathbb{P}(\omega) \\ &= \inf_{n_1, n_2 \in \mathbb{N}} \left\{ \int_W [H_{\mu_{\omega}}(\beta_{\omega}^1 | (\alpha_{n_1}^1)_{\omega}) + H_{\mu_{\omega}}((\alpha_{n_1}^1)_{\omega} | \beta_{\omega}^1)] d\mathbb{P}(\omega) \right. \\ &\quad \left. + \int_{W^c} [H_{\mu_{\omega}}(\beta_{\omega}^2 | (\alpha_{n_2}^2)_{\omega}) + H_{\mu_{\omega}}((\alpha_{n_2}^2)_{\omega} | \beta_{\omega}^2)] d\mathbb{P}(\omega) \right\} \\ &= 0 \text{ (using (6.1))}. \end{aligned}$$

This means that \mathcal{U} is excellent. □

Then we have the following important observation.

Proposition 6.4. *Assume that X is a zero-dimensional space and $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space.*

- (1) *If $\xi \in \mathbf{P}_{\Omega}$ and $\mathcal{V} \in \mathbf{C}_X^o$ then $(\xi \times \mathcal{V})_{\mathcal{E}}$ is excellent.*
- (2) *If $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}^o$ is in the form of $\mathcal{U} = \{(\Omega_i \times U_i)^c : i = 1, \dots, m\}$, $m \in \mathbb{N} \setminus \{1\}$ with $\Omega_i \in \mathcal{F}$ for each $i = 1, \dots, m$ and $\{U_1^c, \dots, U_m^c\} \in \mathbf{C}_X^o$, then \mathcal{U} is good, in fact, there exists $\mathcal{U}' \in \mathbf{C}_{\mathcal{E}}^o$ such that $\mathcal{U}' \succeq \mathcal{U}$, \mathcal{U}' is excellent and $h_{\mu, +}^{(r)}(\mathbf{F}, \mathcal{U}') = h_{\mu, +}^{(r)}(\mathbf{F}, \mathcal{U})$ for each $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$.*

Proof. (1) First, we shall prove the Proposition in the case of $\xi = \{\Omega\}$.

As $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space, there exists an isomorphism $\phi : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (Z, \mathcal{Z}, p)$ between probability spaces, where Z is a zero-dimensional compact metric space and \mathcal{Z} is the completion of \mathcal{B}_Z under p , i.e. there exist $\Omega^* \in \mathcal{F}$, $Z^* \in \mathcal{Z}$ and an invertible measure-preserving transformation $\psi : \Omega^* \rightarrow Z^*$ such that $\mathbb{P}(\Omega^*) = 1 = p(Z^*)$. In fact, it makes no difference to assume $\Omega = \Omega^*$.

Before proceeding, we prove

Claim 6.5. *Set $\psi_* : (\Omega \times X, \mathcal{F} \times \mathcal{B}_X) \rightarrow (Z^* \times X, \mathcal{Z}^* \times \mathcal{B}_X)$, $(\omega, x) \rightarrow (\psi\omega, x)$, where \mathcal{Z}^* is the restriction of \mathcal{Z} over Z^* . Then ψ^* is an invertible bi-measurable map.*

Proof of Claim 6.5. Obviously, $\psi_*^{-1}(C \times D) = \psi_*^{-1}(C) \times D \in \mathcal{F} \times \mathcal{B}_X$ whenever $C \in \mathcal{Z}^*$ and $D \in \mathcal{B}_X$. Whereas, $\{A \in \mathcal{Z}^* \times \mathcal{B}_X : \psi_*^{-1}(A) \in \mathcal{F} \times \mathcal{B}_X\}$ is always a sub- σ -algebra of $\mathcal{Z}^* \times \mathcal{B}_X$, and $\mathcal{Z}^* \times \mathcal{B}_X$ is the smallest σ -algebra containing all $C \times D$, $C \in \mathcal{Z}^*$, $D \in \mathcal{B}_X$, in other words, $\psi_*^{-1}(A) \in \mathcal{F} \times \mathcal{B}_X$ whenever $A \in \mathcal{Z}^* \times \mathcal{B}_X$. This claims the measurability of $\psi_* : (\Omega \times X, \mathcal{F} \times \mathcal{B}_X) \rightarrow (Z^* \times X, \mathcal{Z}^* \times \mathcal{B}_X)$. Similarly, we can show the measurability of ψ_*^{-1} . \square

For each $B \in \mathcal{Z} \times \mathcal{B}_X$, we set

$$B_\psi = \{(\psi^{-1}z, x) : (z, x) \in B \text{ and } z \in Z^*\}.$$

In fact, $B_\psi = \psi_*^{-1}(B \cap (Z^* \times X))$, in particular, $B_\psi \in \mathcal{F} \times \mathcal{B}_X$. Moreover, if $(\Omega \times X, \mathcal{F} \times \mathcal{B}_X, \mu)$ is a probability space, set $\mu_\psi(B) = \mu(B_\psi)$ for each $B \in \mathcal{Z} \times \mathcal{B}_X$, which defines naturally a probability measure over $(Z \times X, \mathcal{B}_Z \times \mathcal{B}_X)$.

Now suppose that $\mathcal{V} = \{V_1, \dots, V_N\}$, $N \in \mathbb{N}$ and set

$$\mathbf{P}_c^*(\Omega \times \mathcal{V}) = \{((A_1)_\psi \cap \mathcal{E}, \dots, (A_N)_\psi \cap \mathcal{E}) : \{A_1, \dots, A_N\} \in \mathbf{P}_c(Z \times \mathcal{V})\}.$$

Observe that, $Z \times X$ is a zero-dimensional compact metric space, by Lemma 6.1, $\mathbf{P}_c(Z \times \mathcal{V})$ is a countable family, and so $\mathbf{P}_c^*(\Omega \times \mathcal{V})$ is also a countable family.

We shall show that $\mathbf{P}_c^*(\Omega \times \mathcal{V})$ satisfies the required properties.

First, by the construction, it is easy to see that, for each $\alpha \in \mathbf{P}_c^*(\Omega \times \mathcal{V})$, $\alpha \in \mathbf{P}_{(\Omega \times \mathcal{V})_\mathcal{E}}$ and α_ω is a clopen partition of \mathcal{E}_ω for \mathbb{P} -a.e. $\omega \in \Omega$. Now if $\beta = \{B_1, \dots, B_N\} \in \mathbf{P}_\mathcal{E}$ satisfies $B_i \subseteq \Omega \times V_i$ for each $i = 1, \dots, N$, it is not hard to obtain some $\beta' = \{B'_1, \dots, B'_N\} \in \mathbf{P}_{Z \times X}$ with $\psi_*(B_i) \subseteq B'_i \subseteq Z \times V_i$ for each $i = 1, \dots, N$. For each $\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E})$, μ may be viewed as a probability measure over $(\Omega \times X, \mathcal{F} \times \mathcal{B}_X)$, and so by Lemma 6.1 for each $\epsilon > 0$ there exists $\alpha' = \{A_1, \dots, A_N\} \in \mathbf{P}_c(Z \times \mathcal{V})$ with

$$H_{\mu_\psi}(\alpha'|\beta') + H_{\mu_\psi}(\beta'|\alpha') < \epsilon.$$

Set $\alpha = \{A_\psi \cap \mathcal{E} : A \in \alpha'\} \in \mathbf{P}_c^*(\Omega \times \mathcal{V})$. As $\mu(\mathcal{E}) = 1$, by the constructions it is easy to check $\mu(B_i) = \mu_\psi(B'_i)$, $\mu((A_i)_\psi \cap \mathcal{E}) = \mu_\psi(A_i)$ and $\mu((A_i)_\psi \cap \mathcal{E} \cap B_j) = \mu_\psi(A_i \cap B'_j)$ for all $i, j = 1, \dots, N$ and so

$$\begin{aligned} H_\mu(\alpha|\beta \vee \mathcal{F}_\mathcal{E}) + H_\mu(\beta|\alpha \vee \mathcal{F}_\mathcal{E}) \\ \leq H_\mu(\alpha|\beta) + H_\mu(\beta|\alpha) = H_{\mu_\psi}(\alpha'|\beta') + H_{\mu_\psi}(\beta'|\alpha') < \epsilon. \end{aligned}$$

This finishes the proof in the case of $\xi = \{\Omega\}$.

Now we shall prove the Proposition for a general $\xi \in \mathbf{P}_\Omega$. In fact,

$$(\xi \times \mathcal{V})_\mathcal{E} = (\xi \times X)_\mathcal{E} \vee (\Omega \times \mathcal{V})_\mathcal{E}.$$

Now from the definition it follows that $(\xi \times X)_\mathcal{E} \in \mathbf{C}_X^\circ$ is excellent (as $\mathbf{P}_{(\xi \times X)_\mathcal{E}} = \{(\xi \times X)_\mathcal{E}\}$) and by the above arguments $(\Omega \times \mathcal{V})_\mathcal{E} \in \mathbf{C}_X^\circ$ is excellent, thus using Lemma 6.3 one obtains that $(\xi \times \mathcal{V})_\mathcal{E}$ is also excellent.

(2) Obviously, in \mathcal{F} there exists disjoint $\Omega'_i \subseteq \Omega_i^c, i = 1, \dots, m$ with $\bigcup_{i=1}^m \Omega'_i = \bigcup_{i=1}^m \Omega_i^c$. Now set $\Omega_0 = \Omega \setminus \bigcup_{i=1}^m \Omega'_i = \bigcap_{i=1}^m \Omega_i$ and

$$\mathcal{U}' = \{(\Omega'_i \times X) \cap \mathcal{E} : i = 1, \dots, m\} \cup \{(\Omega_0 \times U_i^c) \cap \mathcal{E} : i = 1, \dots, m\}.$$

It is easy to see $\mathcal{U}' \in \mathbf{C}_\mathcal{E}^\circ$ and $\mathcal{U}' \succeq \mathcal{U}$. In fact, $\mathcal{U}'_\omega = \mathcal{U}_\omega$ for \mathbb{P} -a.e. $\omega \in \Omega$ and so by Lemma 4.8 one has that $h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}') = h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U})$ for each $\mu \in \mathcal{P}(\mathcal{E}, G)$.

Now with the help of Lemma 6.2 we shall finish our proof by showing that \mathcal{U}' is excellent. In fact, suppose that $\xi = \{\Omega'_i : i = 1, \dots, m\} \cup \{\Omega_0\} \in \mathbf{P}_\Omega$. Then

$$\mathcal{U}' = (\xi \times X)_\mathcal{E} \cap (\Omega_0^c \times X) \cup (\Omega \times \mathcal{V})_\mathcal{E} \cap (\Omega_0 \times X),$$

where $\mathcal{V} = \{U_1^c, \dots, U_m^c\}$, observe that by (1) one has that $(\xi \times X)_\mathcal{E}, (\Omega \times \mathcal{V})_\mathcal{E} \in \mathbf{C}_\mathcal{E}^\circ$ are both excellent, and so using Lemma 6.3 we claim that \mathcal{U}' is excellent. \square

For each $i = 1, 2$, let X_i be a compact metric space with $\mathcal{E}_i \in \mathcal{F} \times \mathcal{B}_{X_i}$ and the family $\mathbf{F}_i = \{(F_i)_{g,\omega} : (\mathcal{E}_i)_\omega \rightarrow (\mathcal{E}_i)_{g\omega} | g \in G, \omega \in \Omega\}$ the corresponding continuous bundle RDS. By a *factor map from \mathbf{F}_1 to \mathbf{F}_2* we mean a measurable map $\pi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ satisfying

- (1) π_ω , the restriction of π over $(\mathcal{E}_1)_\omega$, is a continuous surjection from $(\mathcal{E}_1)_\omega$ to $(\mathcal{E}_2)_\omega$ for \mathbb{P} -a.e. $\omega \in \Omega$ and
- (2) $\pi_{g\omega} \circ (F_1)_{g,\omega} = (F_2)_{g,\omega} \circ \pi_\omega$ for each $g \in G$ and \mathbb{P} -a.e. $\omega \in \Omega$.

In this case, it is obvious that $\pi^{-1}(\mathcal{U}_2) \in \mathbf{P}_{\mathcal{E}_1}$ ($\mathbf{C}_{\mathcal{E}_1}$, $\mathbf{C}_{\mathcal{E}_1}^\circ$, respectively) if $\mathcal{U}_2 \in \mathbf{P}_{\mathcal{E}_2}$ ($\mathbf{C}_{\mathcal{E}_2}$, $\mathbf{C}_{\mathcal{E}_2}^\circ$, respectively). $\mathcal{U}_2 \in \mathbf{C}_{\mathcal{E}_2}^\circ$ is called *factor excellent* (*factor good*, respectively) if there exists such a factor map π with $\pi^{-1}(\mathcal{U}_2)$ excellent (good, respectively).

Let $\mathcal{U} \in \mathbf{C}_\mathcal{E}^\circ$. In general we don't know whether \mathcal{U} is (factor) good, even if X is a zero-dimensional space and $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space. However, we have:

Lemma 6.6. *Let $\mathcal{U} = \{U_1, \dots, U_N\} \in \mathbf{C}_\mathcal{E}^\circ, N \in \mathbb{N}$. Assume that X is a zero-dimensional space. Then there exists $\alpha = \{A_1, \dots, A_N\} \in \mathbf{P}_\mathcal{E}$ such that $\alpha \succeq \mathcal{U}$ and α_ω is a clopen partition of \mathcal{E}_ω for \mathbb{P} -a.e. $\omega \in \Omega$.*

Proof. Say $\pi : \Omega \times X \rightarrow X$ to be the natural projection. Absolutely, we may assume without any difference that \mathcal{E}_ω is a non-empty compact subset of X and $\mathcal{U}_\omega \in \mathbf{C}_{\mathcal{E}_\omega}^\circ$ for each $\omega \in \Omega$.

As X is zero-dimensional, there exists a countable topological basis $\{V_n : n \in \mathbb{N}\}$ of X consisting of clopen subsets (here, we take $V_1 = \emptyset$).

Note that, if I_1, \dots, I_N are N finite disjoint non-empty subsets of \mathbb{N} , and we set

$$\Omega(I_1, \dots, I_N) = \pi((\Omega \times X \setminus \bigcup_{j \in \bigcup_{i=1}^N I_i} V_j) \cap \mathcal{E}) \cup \bigcup_{i=1}^N \pi((\Omega \times \bigcup_{j \in I_i} V_j \setminus U_i) \cap \mathcal{E}),$$

then by Lemma 4.2 one has $\Omega(I_1, \dots, I_N) \in \mathcal{F}$. Moreover, $\omega \notin \Omega(I_1, \dots, I_N)$ if and only if $\mathcal{E}_\omega \subseteq \bigcup_{j \in \bigcup_{i=1}^N I_i} V_j$ and $\bigcup_{j \in I_i} V_j \cap \mathcal{E}_\omega \subseteq (U_i)_\omega$ for each $i = 1, \dots, N$.

Now for any given $\omega \in \Omega$, as $\mathcal{U}_\omega \in \mathbf{C}_{\mathcal{E}_\omega}^\circ$ and X is a zero-dimensional space, there exists $\alpha(\omega) \in \mathbf{P}_{\mathcal{E}_\omega}$ consisting of clopen subsets $A_1(\omega), \dots, A_N(\omega)$ with the property that $A_i(\omega) \subseteq (U_i)_\omega, i = 1, \dots, N$. Furthermore, there exist N finite disjoint non-empty subsets $I_1(\omega), \dots, I_N(\omega) \subseteq \mathbb{N}$ such that $A_i(\omega) = \bigcup_{j \in I_i(\omega)} V_j \cap \mathcal{E}_\omega$ for each $i = 1, \dots, N$. In particular, $\omega \in \Omega(I_1(\omega), \dots, I_N(\omega))^c$.

Thus, there exists a countably family $\{\{I_1^n, \dots, I_N^n\} : n \in \mathbb{N}\}$ of N finite disjoint non-empty subsets of \mathbb{N} and a sequence $\{\Omega_n : n \in \mathbb{N}\} \subseteq \mathcal{F}$ such that $\bigcup_{n \in \mathbb{N}} \Omega_n = \Omega$, $\Omega_n \cap \Omega_m = \emptyset$ whenever $1 \leq n < m$ and $\mathbb{P}(\Omega_n) > 0, \Omega_n \subseteq \Omega(I_1^n, \dots, I_N^n)^c$ for each $n \in \mathbb{N}$. Now set

$$\alpha = \left\{ \bigcup_{n \in \mathbb{N}} (\Omega_n \times \bigcup_{j \in I_i^n} V_j) \cap \mathcal{E} : i = 1, \dots, N \right\}.$$

From the above construction it is not hard to check that α has the claimed properties. This completes the proof. \square

We also have:

Proposition 6.7. *Let $\mathbf{F} = \{F_{g,\omega} : \mathcal{E}_\omega \rightarrow \mathcal{E}_{g\omega} | g \in G, \omega \in \Omega\}$ be a continuous bundle RDS over $(\Omega, \mathcal{F}, \mathbb{P}, G)$. Then there exists a family $\mathbf{F}' = \{F'_{g,\omega} : \mathcal{E}'_\omega \rightarrow \mathcal{E}'_{g\omega} | g \in G, \omega \in \Omega\}$ (with $\mathcal{E}' \in \mathcal{F} \times \mathcal{B}_{X'}$ and X' a compact metric state space), which is a continuous bundle RDS over $(\Omega, \mathcal{F}, \mathbb{P}, G)$, and a factor map $\pi : \mathcal{E}' \rightarrow \mathcal{E}$ from \mathbf{F}' to \mathbf{F} , such that X' is a zero-dimensional space. In fact, π is induced by a continuous surjection from X' to X .*

Proof. It is well known that there exists a continuous surjection $\phi : C \rightarrow X$, where C is a Cantor space. Then G acts naturally on the space C^G with $g' : (c_g)_{g \in G} \mapsto (c_{g'g})_{g \in G}$ whenever $g' \in G$. There is a natural projection

$$\psi : \Omega \times C^G \rightarrow \Omega \times X, (\omega, (c_g)_{g \in G}) \mapsto (\omega, \phi(c_{e_G})).$$

Now we consider $X' = C^G$ which is a zero-dimensional compact metric space and $\mathcal{E}' = \{(\omega, (c_g)_{g \in G}) \in \psi^{-1}(\mathcal{E}) : \phi(c_g) = F_{g,\omega} \phi(c_{e_G}) \text{ for each } g \in G \text{ and any } \omega \in \Omega\}$ with the family $\mathbf{F}' = \{F'_{g,\omega} : \mathcal{E}'_\omega \rightarrow \mathcal{E}'_{g\omega} | g \in G, \omega \in \Omega\}$ given by

$$F'_{g',\omega} : \mathcal{E}'_\omega \ni (c_g)_{g \in G} \mapsto (c_{g'g})_{g \in G}, g' \in G, \omega \in \Omega.$$

The map $\pi : \mathcal{E}' \rightarrow \mathcal{E}$ is defined naturally by $(\omega, (c_g)_{g \in G}) \mapsto (\omega, \phi(c_{e_G}))$, and is clearly well-defined. In the following we shall check step by step that $X', \mathcal{E}', \mathbf{F}'$ and π as constructed satisfy the required properties.

- (1) The family $\mathbf{F}' = \{F'_{g,\omega} : \mathcal{E}'_\omega \rightarrow \mathcal{E}'_{g\omega} | g \in G, \omega \in \Omega\}$, which is well defined naturally, is a continuous bundle RDS over $(\Omega, \mathcal{F}, \mathbb{P}, G)$: first, for the map

$$\psi_G : \Omega \times C^G \rightarrow \Omega \times X^G, (\omega, (c_g)_{g \in G}) \mapsto (\omega, (\phi c_g)_{g \in G})$$

which is obviously measurable, $\mathcal{E}' = \psi_G^{-1}(\mathcal{E}_G)$, where

$$\mathcal{E}_G = \{(\omega, (x_g)_{g \in G}) : (\omega, x_{e_G}) \in \mathcal{E}, x_g = F_{g,\omega} x_{e_G} \text{ for each } g \in G \text{ and any } \omega \in \Omega\},$$

then $\mathcal{E}' \in \mathcal{F} \times \mathcal{B}_{X'}$ follows from $\mathcal{E}_G \in \mathcal{F} \times \mathcal{B}_{X^G}$. Secondly, the measurability of

$$(\omega, (c_g)_{g \in G}) \in \mathcal{E}' \mapsto F'_{g',\omega}((c_g)_{g \in G}) = (c_{g'g})_{g \in G}$$

for fixed $g' \in G$ and the equality $F'_{g_2, g_1 \omega} \circ F'_{g_1, \omega} = F'_{g_2 g_1, \omega}$ for each $\omega \in \Omega$ and all $g_1, g_2 \in G$ are easy to see. Finally, it is not hard to check that

- $\emptyset \neq \mathcal{E}'_\omega \subseteq X'$ is a compact subset and $F'_{g,\omega}$ is continuous for each $g \in G$. We have shown that the family \mathbf{F}' is a continuous bundle RDS over $(\Omega, \mathcal{F}, \mathbb{P}, G)$.
- (2) π is a factor map from \mathcal{E}' to \mathcal{E} : in fact, let $\omega \in \Omega$, obviously $\pi_\omega : \mathcal{E}'_\omega \rightarrow \mathcal{E}_\omega$ is a continuous surjection; now let $g' \in G$, if $(\omega, (c_g)_{g \in G}) \in \mathcal{E}'$ then

$$\pi_{g'\omega} \circ F'_{g',\omega}((c_g)_{g \in G}) = \pi_{g'\omega}((c_{g'g})_{g \in G}) = \phi(c_{g'}) = F_{g',\omega} \circ \phi(c_{eG}) = F_{g',\omega} \circ \pi_\omega((c_g)_{g \in G}),$$

which establishes the identity $\pi_{g'\omega} \circ F'_{g',\omega} = F_{g',\omega} \circ \pi_\omega$.

It is clear that π is induced by the continuous surjection $X' \rightarrow X, (c_g)_{g \in G} \mapsto \phi(c_{eG})$. This completes the proof. \square

Suppose that the family $\mathbf{F}_i = \{(F_i)_{g,\omega} : (\mathcal{E}_i)_\omega \rightarrow (\mathcal{E}_i)_{g\omega} | g \in G, \omega \in \Omega\}$ is a continuous bundle RDS over $(\Omega, \mathcal{F}, \mathbb{P}, G)$, $i = 1, 2$ and $\pi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ a factor map from \mathbf{F}_1 to \mathbf{F}_2 . π naturally induces a map from $\mathcal{P}_\mathbb{P}(\mathcal{E}_1)$ to $\mathcal{P}_\mathbb{P}(\mathcal{E}_2)$, which is still denoted by π without any ambiguity.

It is now almost a direct consequence that:

Lemma 6.8. *Suppose that for $i = 1, 2$ the family $\mathbf{F}_i = \{(F_i)_{g,\omega} : (\mathcal{E}_i)_\omega \rightarrow (\mathcal{E}_i)_{g\omega} | g \in G, \omega \in \Omega\}$ is a continuous bundle RDS over $(\Omega, \mathcal{F}, \mathbb{P}, G)$ with corresponding compact metric state space X_i . Assume that $\pi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is a factor map from \mathbf{F}_1 to \mathbf{F}_2 , $\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E}_1, G)$, $\alpha \in \mathbf{P}_{\mathcal{E}_2}$, $\mathcal{U} \in \mathbf{C}_{\mathcal{E}_2}$ and $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}_2}^1(\Omega, C(X_2))$ is a sub-additive G -invariant family. Then*

- (1) *If the sequence $\{\eta_n : n \in \mathbb{N}\}$ converges to η in $\mathcal{P}_\mathbb{P}(\mathcal{E}_1)$ then the sequence $\{\pi\eta_n : n \in \mathbb{N}\}$ converges to $\pi\eta$ in $\mathcal{P}_\mathbb{P}(\mathcal{E}_2)$. In other words, the map $\pi : \mathcal{P}_\mathbb{P}(\mathcal{E}_1) \rightarrow \mathcal{P}_\mathbb{P}(\mathcal{E}_2)$ is continuous.*
- (2) *$\pi\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E}_2, G)$.*
- (3) *$\mathbf{D} \circ \pi \doteq \{d_F \circ \pi : F \in \mathcal{F}_G\}$ is a sub-additive G -invariant family in $\mathbf{L}_{\mathcal{E}_1}^1(\Omega, C(X_1))$. Moreover, if \mathbf{D} is monotone then $\mathbf{D} \circ \pi$ is also monotone.*
- (4) *$h_\mu^{(r)}(\mathbf{F}_1, \pi^{-1}\alpha) = h_{\pi\mu}^{(r)}(\mathbf{F}_2, \alpha)$ and so $h_{\mu,+}^{(r)}(\mathbf{F}_1, \pi^{-1}\mathcal{U}) \leq h_{\pi\mu,+}^{(r)}(\mathbf{F}_2, \mathcal{U})$.*
- (5) *$h_\mu^{(r)}(\mathbf{F}_1, \pi^{-1}\mathcal{U}) = h_{\pi\mu}^{(r)}(\mathbf{F}_2, \mathcal{U})$ and so $h_{\pi\mu}^{(r)}(\mathbf{F}_1) \geq h_{\pi\mu}^{(r)}(\mathbf{F}_2)$.*
- (6) *For each $F \in \mathcal{F}_G$ and for any $\omega \in \Omega$,*

$$P_{\mathcal{E}_1}(\omega, \mathbf{D} \circ \pi, F, \pi^{-1}\mathcal{U}, \mathbf{F}_1) = P_{\mathcal{E}_2}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}_2).$$

Hence if \mathbf{D} is monotone then $P_{\mathcal{E}_1}(\mathbf{D} \circ \pi, \pi^{-1}\mathcal{U}, \mathbf{F}_1) = P_{\mathcal{E}_2}(\mathbf{D}, \mathcal{U}, \mathbf{F}_2)$. In particular, $h_{top}^{(r)}(\mathbf{F}_1, \pi^{-1}\mathcal{U}) = h_{top}^{(r)}(\mathbf{F}_2, \mathcal{U})$. As a consequence,

$$P_{\mathcal{E}_1}(\mathbf{D} \circ \pi, \mathbf{F}_1) \geq P_{\mathcal{E}_2}(\mathbf{D}, \mathbf{F}_2) \text{ and } h_{top}^{(r)}(\mathbf{F}_1) \geq h_{top}^{(r)}(\mathbf{F}_2).$$

Proof. The first four statements are easy to check; we prove the last two.

In fact, the last item follows from (5.5) and the fact of $\mathbf{P}((\pi^{-1}\mathcal{U})_F) = \pi^{-1}\mathbf{P}(\mathcal{U}_F) \doteq \{\{\pi^{-1}B : B \in \beta\} : \beta \in \mathbf{P}(\mathcal{U}_F)\}$ for each $F \in \mathcal{F}_G$.

As for (5), suppose that $d\mu(\omega, x) = d\mu_\omega(x)d\mathbb{P}(\omega)$ is the disintegration of μ over $\mathcal{F}_{\mathcal{E}_1}$. Then it is not hard to check that $d(\pi\mu)(\omega, y) = d(\pi_\omega\mu_\omega)(y)d\mathbb{P}(\omega)$ is the

disintegration of $\pi\mu$ over $\mathcal{F}_{\mathcal{E}_2}$. Hence for each $F \in \mathcal{F}_G$,

$$\begin{aligned}
(6.2) \quad & H_\mu((\pi^{-1}\mathcal{U})_F | \mathcal{F}_{\mathcal{E}_1}) \\
&= \int_{\Omega} H_{\mu_\omega}(((\pi^{-1}\mathcal{U})_F)_\omega) d\mathbb{P}(\omega) \text{ (using (4.5))} \\
&= \int_{\Omega} \inf_{\beta(\omega) \in \mathbf{P}(((\pi^{-1}\mathcal{U})_F)_\omega)} H_{\mu_\omega}(\beta(\omega)) d\mathbb{P}(\omega) \text{ (using (3.2))} \\
&= \int_{\Omega} \inf_{\alpha \in \mathbf{P}((\pi^{-1}\mathcal{U})_F)} H_{\mu_\omega}(\alpha_\omega) d\mathbb{P}(\omega) \text{ (using Lemma 5.2)} \\
&= \int_{\Omega} \inf_{\beta \in \mathbf{P}(\mathcal{U}_F)} H_{\mu_\omega}((\pi^{-1}\beta)_\omega) d\mathbb{P}(\omega) \text{ (as } \mathbf{P}((\pi^{-1}\mathcal{U})_F) = \pi^{-1}\mathbf{P}(\mathcal{U}_F))} \\
&= \int_{\Omega} \inf_{\beta \in \mathbf{P}(\mathcal{U}_F)} H_{\pi_\omega \mu_\omega}(\beta_\omega) d\mathbb{P}(\omega) \\
&= H_{\pi\mu}(\mathcal{U}_F | \mathcal{F}_{\mathcal{E}_2}) \text{ (by a reasoning similar to (6.2)),}
\end{aligned}$$

and so $h_\mu^{(r)}(\mathbf{F}_1, \pi^{-1}\mathcal{U}) = h_{\pi\mu}^{(r)}(\mathbf{F}_2, \mathcal{U})$. This finishes our proof. \square

By Proposition 6.4 and Proposition 6.7, one has:

Theorem 6.9. *Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space.*

- (1) *If $\xi \in \mathbf{P}_\Omega$ and $\mathcal{V} \in \mathbf{C}_X^\circ$ then $(\xi \times \mathcal{V})_\mathcal{E}$ is factor excellent.*
- (2) *If $\mathcal{U} \in \mathbf{C}_\mathcal{E}^\circ$ has the form $\mathcal{U} = \{(\Omega_i \times U_i)^c : i = 1, \dots, n\}$, $n \in \mathbb{N} \setminus \{1\}$ with $\Omega_i \in \mathcal{F}, i = 1, \dots, n$ and $\{U_1^c, \dots, U_n^c\} \in \mathbf{C}_X^\circ$, then \mathcal{U} is factor good.*

By Lemma 6.1 and Proposition 6.7, one has:

Theorem 6.10. *Assume that Ω is a zero-dimensional compact metric space with $\mathcal{F} = \mathcal{B}_\Omega$. Then each member of $\mathbf{C}_\mathcal{E}^{t,o}$ is factor excellent.*

We end this section with the following nice property of a factor good cover.

A generalized real-valued function f defined on a compact space Z is called *upper semi-continuous* (u.s.c.) if one of the following equivalent conditions holds:

- (1) $\limsup_{z' \rightarrow z} f(z') \leq f(z)$ for each $z \in Z$.
- (2) for each $r \in \mathbb{R}$, the set $\{z \in Z : f(z) \geq r\} \subseteq Z$ is closed.

Notice that the infimum of any family of u.s.c. functions is again u.s.c., and similarly both the sum and the supremum of finitely many u.s.c. functions are u.s.c.

It follows that:

Proposition 6.11. *Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space. If $\mathcal{U} \in \mathbf{C}_\mathcal{E}^\circ$ is factor good then both $h_{\bullet}^{(r)}(\mathbf{F}, \mathcal{U}) : \mathcal{P}_\mathbb{P}(\mathcal{E}, G) \rightarrow \mathbb{R}, \mu \mapsto h_\mu^{(r)}(\mathbf{F}, \mathcal{U})$ and $h_{\bullet,+}^{(r)}(\mathbf{F}, \mathcal{U}) : \mathcal{P}_\mathbb{P}(\mathcal{E}, G) \rightarrow \mathbb{R}, \mu \mapsto h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U})$ are u.s.c. functions.*

Proof. As $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space, by Proposition 4.7, we only need check the property of u.s.c. for the function $h_{\bullet,+}^{(r)}(\mathbf{F}, \mathcal{U}) : \mathcal{P}_\mathbb{P}(\mathcal{E}, G) \rightarrow \mathbb{R}, \mu \mapsto h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U})$.

First, we prove the proposition in the case that \mathcal{U} is good. By assumption, there exists a sequence $\{\alpha_n : n \in \mathbb{N}\} \subseteq \mathbf{P}_\mathcal{U}$ satisfying:

- (1) For each $n \in \mathbb{N}$, $(\alpha_n)_\omega$ is a clopen partition of \mathcal{E}_ω for \mathbb{P} -a.e. $\omega \in \Omega$ and
- (2) For each $\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E}, G)$, $h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) = \inf_{n \in \mathbb{N}} h_\mu^{(r)}(\mathbf{F}, \alpha_n)$.

By the construction of the sequence $\{\alpha_n : n \in \mathbb{N}\} \subseteq \mathbf{P}_{\mathcal{U}}$ and using Proposition 4.5, one sees that for each $F \in \mathcal{F}_G$ and any $n \in \mathbb{N}$, the function

$$H_{\bullet}((\alpha_n)_F | \mathcal{F}_{\mathcal{E}}) : \mathcal{P}_{\mathbb{P}}(\mathcal{E}) \rightarrow \mathbb{R}, \mu \mapsto H_{\mu}((\alpha_n)_F | \mathcal{F}_{\mathcal{E}})$$

is u.s.c. It follows that the function

$$h_{\bullet}^{(r)}(\mathbf{F}, \alpha_n) : \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G) \rightarrow \mathbb{R}, \mu \mapsto h_{\mu}^{(r)}(\mathbf{F}, \alpha_n)$$

is also u.s.c. for each $n \in \mathbb{N}$ (using (3.3)), which implies that the function

$$h_{\bullet,+}^{(r)}(\mathbf{F}, \mathcal{U}) : \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G) \rightarrow \mathbb{R}, \mu \mapsto h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) = \inf_{n \in \mathbb{N}} h_{\mu}^{(r)}(\mathbf{F}, \alpha_n)$$

is u.s.c., as it is the infimum of a family of u.s.c. functions.

For the general case, our assumptions imply that there exists a continuous bundle RDS $\mathbf{F}' = \{F'_{g,\omega} : \mathcal{E}'_{\omega} \rightarrow \mathcal{E}'_{g\omega} | g \in G, \omega \in \Omega\}$ (with $\mathcal{E}' \in \mathcal{F} \times \mathcal{B}_{X'}$ and X' a compact metric state space) and a factor map $\pi : \mathcal{E}' \rightarrow \mathcal{E}$ from \mathbf{F}' to \mathbf{F} such that $\pi^{-1}\mathcal{U}$ is good. By the above arguments, the function $h_{\bullet,+}^{(r)}(\mathbf{F}', \pi^{-1}\mathcal{U}) : \mathcal{P}_{\mathbb{P}}(\mathcal{E}', G) \rightarrow \mathbb{R}, \mu' \mapsto h_{\mu',+}^{(r)}(\mathbf{F}', \pi^{-1}\mathcal{U})$ is u.s.c. As $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space, we may apply Proposition 4.7 and Lemma 6.8 to deduce:

$$h_{\pi\mu',+}^{(r)}(\mathbf{F}, \mathcal{U}) = h_{\pi\mu'}^{(r)}(\mathbf{F}, \mathcal{U}) = h_{\mu'}^{(r)}(\mathbf{F}', \pi^{-1}\mathcal{U}) = h_{\mu',+}^{(r)}(\mathbf{F}', \pi^{-1}\mathcal{U}).$$

For each $\mu' \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}', G)$. Thus, for each $r \in \mathbb{R}$ (recall that $\pi\mathcal{P}_{\mathbb{P}}(\mathcal{E}', G) = \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ [53, Proposition 2.5] and $\pi : \mathcal{P}_{\mathbb{P}}(\mathcal{E}', G) \rightarrow \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ is continuous by Lemma 6.8),

$$\{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G) : h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) \geq r\} = \pi(\{\mu' \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}', G) : h_{\mu',+}^{(r)}(\mathbf{F}', \pi^{-1}\mathcal{U}) \geq r\})$$

is also a closed subset, which finishes our proof. \square

7. A VARIATIONAL PRINCIPLE FOR LOCAL FIBER TOPOLOGICAL PRESSURE

In this section we present our main result, Theorem 7.1. As its proof is somewhat technical and complicated, we postpone it to next section, and in this section we give the statement, some remarks and direct applications of it.

Here is our main result.

Theorem 7.1. *Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a monotone sub-additive G -invariant family and $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}^{\circ}$. Assume that \mathbf{D} satisfies:*

- for any given sequence $\{\nu_n : n \in \mathbb{N}\} \subseteq \mathcal{P}_{\mathbb{P}}(\mathcal{E})$, set $\mu_n = \frac{1}{|F_n|} \sum_{g \in F_n} g\nu_n$ for each $n \in \mathbb{N}$, then there always exists some sub-sequence $\{n_j : j \in \mathbb{N}\} \subseteq \mathbb{N}$ such that the sequence $\{\mu_{n_j} : j \in \mathbb{N}\}$ converges to some $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$ (and
- (♠) so $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$) and

$$\limsup_{j \rightarrow \infty} \frac{1}{|F_{n_j}|} \int_{\mathcal{E}} d_{F_{n_j}}(\omega, x) d\nu_{n_j}(\omega, x) \leq \mu(\mathbf{D}).$$

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space and \mathcal{U} is factor good then

$$P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) = \max_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D})] = \max_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D})],$$

moreover, combining with Theorem 4.11 and Theorem 6.9 one has

$$P_{\mathcal{E}}(\mathbf{D}, \mathbf{F}) = \sup_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}) + \mu(\mathbf{D})],$$

in particular,

$$h_{top}^{(r)}(\mathbf{F}, \mathcal{U}) = \max_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) = \max_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}),$$

$$h_{top}^{(r)}(\mathbf{F}) = \sup_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} h_{\mu}^{(r)}(\mathbf{F}).$$

Remark 7.2. This result may be viewed as a local version of [48, Theorem 2.1] in the general case of a continuous bundle RDS. Moreover, $h_{\mu}^{(r)}(\mathbf{F}) + \mu(\mathbf{D})$ may be viewed as a general definition of measure-theoretic pressure in our setting. We should note that [74] provides another possible direct but more complicated definition of measure-theoretic pressure in the setting of topological dynamical systems.

Remark 7.3. In fact, by the proof given in §8 we will obtain a local version of [56, Variational Principle 5.2.7] (see [56, Theorem 5.2.8 and Theorem 5.2.13]) in our more general setting. Specifically, let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a family satisfying (\spadesuit) and $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}^o$, here, in the assumption of (\clubsuit) for $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ we use

$$\limsup_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\mathcal{E}} d_{F_n}(\omega, x) d\mu(\omega, x)$$

to replace $\mu(\mathbf{D})$, if $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space and \mathcal{U} is factor good then

$$(7.1) \quad \limsup_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\Omega} \log P_{\mathcal{E}}(\omega, \mathbf{D}, F_n, \mathcal{U}, \mathbf{F}) d\mathbb{P}(\omega) \\ = \max_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}) + \limsup_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\mathcal{E}} d_{F_n}(\omega, x) d\mu(\omega, x)].$$

Observe that $P_{\mathcal{E}}(\omega, \mathbf{D}, F_n, \mathcal{U}, \mathbf{F})$ can be introduced similarly. In addition, it is not hard to obtain [56, Variational Principle 5.2.7] from (7.1). In particular, for each $f \in \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$, obviously \mathbf{D}^f is a family in $\mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ satisfying the assumption of (\clubsuit) , and so in the case that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space and $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}^o$ is factor good, we obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\Omega} \log P_{\mathcal{E}}(\omega, \mathbf{D}^f, F_n, \mathcal{U}, \mathbf{F}) d\mathbb{P}(\omega) \\ = \max_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) + \int_{\mathcal{E}} f(\omega, x) d\mu(\omega, x)] \\ = \max_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}) + \int_{\mathcal{E}} f(\omega, x) d\mu(\omega, x)]$$

and (using Theorem 4.11 and Theorem 6.9)

$$\sup_{\mathcal{V} \in \mathbf{C}_X^o} \limsup_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\Omega} \log P_{\mathcal{E}}(\omega, \mathbf{D}^f, F_n, (\Omega \times \mathcal{V})_{\mathcal{E}}, \mathbf{F}) d\mathbb{P}(\omega) \\ = \sup_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}) + \int_{\mathcal{E}} f(\omega, x) d\mu(\omega, x)].$$

Remark 7.4. We believe that Theorem 7.1 holds for a general $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}^o$, but we have not so far been able to prove it in full generality. In fact, inspired by Proposition 6.7 (and Theorem 6.9, Theorem 6.10) it seems possible to prove that each $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}^o$ is factor good and so Theorem 7.1 will hold for all $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}^o$.

Remark 7.5. Combined with Proposition 4.5, we believe that a monotone sub-additive G -invariant family $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_\mathcal{E}^1(\Omega, C(X))$ always satisfies the assumption (\spadesuit) , and if this was the case, we would be able to prove the Theorem in generality. We shall discuss this assumption in §9 and in §10 we will show that it holds for some special cases.

Remark 7.6. In fact, if $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_\mathcal{E}^1(\Omega, C(X))$ is just a sub-additive G -invariant family satisfying (\spadesuit) (which need not to be monotone), and if, in addition, there exists a finite constant $C \in \mathbb{R}_+$ such that $\mathbf{D}' = \{d'_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_\mathcal{E}^1(\Omega, C(X))$ is a monotone sub-additive G -invariant family, where $d'_F = d_F + |F|C$ for each $F \in \mathcal{F}_G$, then we can introduce $P_\mathcal{E}(\mathbf{D}, \mathcal{U}, \mathbf{F})$, $P_\mathcal{E}(\mathbf{D}, \mathbf{F})$ and $\mu(\mathbf{D})$ similarly for each $\mathcal{U} \in \mathbf{C}_\mathcal{E}$ and any $\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E}, G)$. It is easy to check that the family \mathbf{D}' also satisfies (\spadesuit) . Hence in the case that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space and $\mathcal{U} \in \mathbf{C}_\mathcal{E}^\circ$ is factor good, we may apply Theorem 7.1 to \mathbf{D}' and \mathcal{U} , and by standard arguments we obtain

$$P_\mathcal{E}(\mathbf{D}, \mathcal{U}, \mathbf{F}) = \max_{\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E}, G)} [h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D})] = \max_{\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E}, G)} [h_\mu^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D})]$$

and

$$P_\mathcal{E}(\mathbf{D}, \mathbf{F}) = \sup_{\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E}, G)} [h_\mu^{(r)}(\mathbf{F}) + \mu(\mathbf{D})].$$

As a direct corollary, we can strengthen Lemma 5.7 as follows.

Proposition 7.7. Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_\mathcal{E}^1(\Omega, C(X))$ be a monotone sub-additive G -invariant family satisfying the assumption of (\spadesuit) . If $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space then

$$\sup_{\mathbb{P}}(\mathbf{D}) = \max_{\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E}, G)} \mu(\mathbf{D}).$$

Proof. Observe that $\{\mathcal{E}\} = (\Omega \times \{X\})_\mathcal{E} \in \mathbf{C}_\mathcal{E}^\circ$ is excellent, and so by Theorem 7.1 one has

$$P_\mathcal{E}(\mathbf{D}, \{\mathcal{E}\}, \mathbf{F}) = \max_{\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E}, G)} [h_\mu^{(r)}(\mathbf{F}, \{\mathcal{E}\}) + \mu(\mathbf{D})].$$

It is easy to see that $h_{\text{top}}^{(r)}(\mathbf{F}, \{\mathcal{E}\}) = 0$ and $h_\mu^{(r)}(\mathbf{F}, \{\mathcal{E}\}) = 0$ for each $\mu \in \mathcal{P}_\mathbb{P}(\mathcal{E}, G)$, and so by Proposition 5.8 we have the conclusion. \square

Observe that we may deduce a result analogous to Remark 7.3.

The concept of a principal extension was firstly introduced and studied by Ledrappier in [47]. It plays an important role in relative entropy theory. Inspired by this, we can also introduce it in our setting.

Let the family $\mathbf{F}_i = \{(F_i)_{g,\omega} : (\mathcal{E}_i)_\omega \rightarrow (\mathcal{E}_i)_{g\omega} | g \in G, \omega \in \Omega\}$ be a continuous bundle RDS over $(\Omega, \mathcal{F}, \mathbb{P}, G)$ with X_i the corresponding compact metric state space, $i = 1, 2$ and $\pi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ a factor map from \mathbf{F}_1 to \mathbf{F}_2 . π is called *principal* if $h_{\mu_1}^{(r)}(\mathbf{F}_1) = h_{\pi\mu_1}^{(r)}(\mathbf{F}_2)$ for each $\mu_1 \in \mathcal{P}_\mathbb{P}(\mathcal{E}_1, G)$.

Before proceeding, we also need the following result.

Lemma 7.8. Let the family $\mathbf{F}_i = \{(F_i)_{g,\omega} : (\mathcal{E}_i)_\omega \rightarrow (\mathcal{E}_i)_{g\omega} | g \in G, \omega \in \Omega\}$ be a continuous bundle RDS over $(\Omega, \mathcal{F}, \mathbb{P}, G)$ with X_i the corresponding compact metric state space, $i = 1, 2$ and $\pi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ a factor map from \mathbf{F}_1 to \mathbf{F}_2 . Assume that $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_\mathcal{E}^1(\Omega, C(X_2))$ satisfies the assumption of (\spadesuit) with respect to \mathbf{F}_2 . Then $\mathbf{D} \circ \pi$ satisfies the assumption of (\spadesuit) with respect to \mathbf{F}_1 .

Proof. Let $\{\nu_n : n \in \mathbb{N}\} \subseteq \mathcal{P}_{\mathbb{P}}(\mathcal{E}_1)$ be a given sequence and set $\mu_n = \frac{1}{|F_n|} \sum_{g \in F_n} g\nu_n$ for each $n \in \mathbb{N}$. As \mathbf{D} satisfies the assumption of (\spadesuit) with respect to \mathbf{F}_2 , then there always exists some sub-sequence $\{n_j : j \in \mathbb{N}\} \subseteq \mathbb{N}$ such that the sequence $\{\pi\mu_{n_j} : j \in \mathbb{N}\}$ converges to some $\mu' \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}_2, G)$ and

$$(7.2) \quad \limsup_{j \rightarrow \infty} \frac{1}{|F_{n_j}|} \int_{\mathcal{E}_2} d_{F_{n_j}}(\omega, x) d\pi\nu_{n_j}(\omega, x) \leq \mu'(\mathbf{D}).$$

Note that by Proposition 4.5 we may assume that $\{\mu_{n_j} : j \in \mathbb{N}\}$ converges to some $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}_1, G)$ (by selecting a sub-sequence of $\{n_j : j \in \mathbb{N}\}$ if necessary). Obviously, $\pi\mu = \mu'$ and then (7.2) can be restated as:

$$\limsup_{j \rightarrow \infty} \frac{1}{|F_{n_j}|} \int_{\mathcal{E}_1} d_{F_{n_j}} \circ \pi(\omega, x) d\nu_{n_j}(\omega, x) \leq \mu(\mathbf{D} \circ \pi).$$

That is, $\mathbf{D} \circ \pi$ satisfies the assumption of (\spadesuit) with respect to \mathbf{F}_1 . \square

Now given a factor map between continuous bundle RDSs $\pi : \mathbf{F}_1 \rightarrow \mathbf{F}_2$ it was proved $\pi\mathcal{P}_{\mathbb{P}}(\mathcal{E}_1, G) = \mathcal{P}_{\mathbb{P}}(\mathcal{E}_2, G)$ [53, Proposition 2.5]. Thus, by the definition, Theorem 7.1 and Lemma 7.8 one has:

Proposition 7.9. *Let the family $\mathbf{F}_i = \{(F_i)_{g,\omega} : (\mathcal{E}_i)_{\omega} \rightarrow (\mathcal{E}_i)_{g\omega} | g \in G, \omega \in \Omega\}$ be a continuous bundle RDS over $(\Omega, \mathcal{F}, \mathbb{P}, G)$ with X_i the corresponding compact metric state space, $i = 1, 2$ and $\pi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ a factor map from \mathbf{F}_1 to \mathbf{F}_2 . Assume that $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}_2}^1(\Omega, C(X_2))$ is a monotone sub-additive G -invariant family satisfying the assumption of (\spadesuit) with respect to \mathbf{F}_2 . If π is principal and $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space then*

$$P_{\mathcal{E}_2}(\mathbf{D}, \mathbf{F}_2) = P_{\mathcal{E}_1}(\mathbf{D} \circ \pi, \mathbf{F}_1), \text{ particularly, } h_{\text{top}}^{(r)}(\mathbf{F}_2) = h_{\text{top}}^{(r)}(\mathbf{F}_1).$$

Remark 7.10. *Given a factor map between continuous bundle RDSs $\pi : \mathbf{F}_1 \rightarrow \mathbf{F}_2$ over $(\Omega, \mathcal{F}, \mathbb{P}, G)$, for each $\mu_1 \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}_1, G)$ we see that π may be viewed as a given G -invariant sub- σ -algebra \mathcal{C} of an MDS $(\mathcal{E}_1, (\mathcal{F} \times \mathcal{B}_{X_1}) \cap \mathcal{E}_1, \mu_1, G)$. If the state space $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space, then a special case of π being a principal extension is $h_{\mu_1}(G, \mathcal{E}_1 | \mathcal{C}) = 0$ for each $\mu_1 \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}_1, G)$, as the well-known Abramov-Rokhlin entropy addition formula states*

$$h_{\mu_1}^{(r)}(\mathbf{F}_1) \leq h_{\pi\mu_1}^{(r)}(\mathbf{F}_2) + h_{\mu_1}(G, \mathcal{E}_1 | \mathcal{C}),$$

in the notation of our setting (see Proposition 3.10). Thus, by Proposition 3.16 one sees that [53, Theorem 2.3] is just a very special case of a principal extension and so [53, Theorem 2.3] follows directly from Proposition 7.9 (and its variants, see Remark 7.3 and §10).

8. PROOF OF THEOREM 7.1

In this section, we present the technical and complicated proof of Theorem 7.1 following the ideas of [35, 37, 55, 74] and the references therein.

In fact, using Proposition 4.7 and Proposition 5.6, we can deduce Theorem 7.1 from the following result.

Proposition 8.1. *Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a monotone sub-additive G -invariant family satisfying the assumption of (\spadesuit) and $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}^{\circ}$. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space and \mathcal{U} is factor good then, for some $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$,*

$$h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D}) \geq P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}).$$

Before proceeding, we need:

Lemma 8.2. *Let (Γ, \mathcal{T}) be a measurable space, X a Polish space, $p : \Gamma \rightarrow X$ a measurable map and $\alpha \in \mathbf{P}_{\Gamma \times X}$. Then*

$$\bigcup_{\tau \in \Gamma} \{\tau\} \times \alpha_\tau(p(\tau)) \in \mathcal{T} \times \mathcal{B}_X.$$

Here, both $\mathbf{P}_{\Gamma \times X}$ and $\alpha_\tau, \tau \in \Gamma$ are introduced similarly as in previous sections.

Proof. Suppose that $\pi : \Gamma \times X \rightarrow \Gamma$ is the natural projection and set $B = \{(\tau, p(\tau)) : \tau \in \Gamma\}$. Then $B \in \mathcal{T} \times \mathcal{B}_X$. It is clear that there exist distinct atoms $A_1, \dots, A_n, n \in \mathbb{N}$ from α such that $B \subseteq \bigcup_{i=1}^n A_i$ and $B \cap A_i \neq \emptyset$ for each $i = 1, \dots, n$. In fact, for each $i = 1, \dots, n$, set $C_i = \pi(B \cap A_i)$. Then $\bigcup_{k=1}^n C_k = \Gamma, C_i \in \mathcal{T}$ (using Lemma 4.2) and $C_i \cap C_j = \emptyset$ if $1 \leq i \neq j \leq n$, and so $\{C_1, \dots, C_n\} \in \mathbf{P}_\Gamma$ (here, \mathbf{P}_Γ is introduced as in previous sections). Moreover,

$$\bigcup_{\tau \in \Gamma} \{\tau\} \times \alpha_\tau(p(\tau)) = \bigcup_{i=1}^n \bigcup_{\tau \in C_i} \{\tau\} \times \alpha_\tau(p(\tau)) = \bigcup_{i=1}^n [(C_i \times X) \cap A_i] \in \mathcal{T} \times \mathcal{B}_X.$$

This completes the proof. \square

We also need the following selection lemma, which is a random variation of [74, Lemma 3.1]. It plays a key role in the establishment of Theorem 7.1.

Lemma 8.3. *Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_\mathcal{E}^1(\Omega, C(X))$ and $\mathcal{U} \in \mathbf{C}_\mathcal{E}$. Assume that $\alpha_k \in \mathbf{P}_\mathcal{E}$ satisfies $\alpha_k \succeq \mathcal{U}$ for each $1 \leq k \leq K$, where $K \in \mathbb{N}$. Then for each $F \in \mathcal{F}_G$ there exists a family of finite subsets $B_{F,\omega} \subseteq \mathcal{E}_\omega, \omega \in \Omega$ such that*

(1) *For $B_F \doteq \{(\omega, x) : \omega \in \Omega, x \in B_{F,\omega}\}$,*

$$\sum_{x \in B_{F,\omega}} e^{d_F(\omega, x)} > \frac{1}{K} \left[\inf_{\beta(\omega) \in \mathbf{P}_{\mathcal{E}_\omega}, \beta(\omega) \succeq (\mathcal{U}_F)_\omega} \sum_{B \in \beta(\omega)} \sup_{x \in B} e^{d_F(\omega, x)} - \frac{1}{2} e^{-\|d_F(\omega)\|_\infty} \right],$$

(2) *The family depends measurably on $\omega \in \Omega$ in the sense of $B_F \in \mathcal{F} \times \mathcal{B}_X$ and*

(3) *Each atom of $((\alpha_k)_F)_\omega$ contains at most one point from $B_{F,\omega}, 1 \leq k \leq K$.*

Proof. Let $\pi : \Omega \times X \rightarrow \Omega$ be the natural projection. Set $\mathcal{E}_0 = \mathcal{E}$. By Lemma 4.1 there exists a measurable map $p_1 : \Omega \rightarrow X$ such that $(\omega, p_1(\omega)) \in \mathcal{E}_0$ for each $\omega \in \pi(\mathcal{E}_0)$ (it makes no any difference to assume that $\pi(\mathcal{E}_0) = \Omega$) and

$$e^{d_F(\omega, p_1(\omega))} \geq \sup_{x \in (\mathcal{E}_0)_\omega} e^{d_F(\omega, x)} - \frac{1}{2^{1+1}K} e^{-\|d_F(\omega)\|_\infty}.$$

Note that by Lemma 8.2, for each $k = 1, \dots, K$,

$$\bigcup_{\omega \in \Omega} \{\omega\} \times ((\alpha_k)_F)_\omega(p_1(\omega)) \in \mathcal{F} \times \mathcal{B}_X,$$

and so

$$\mathcal{E}_1 \doteq \mathcal{E}_0 \setminus \bigcup_{k=1}^K \bigcup_{\omega \in \pi(\mathcal{E}_0)} \{\omega\} \times ((\alpha_k)_F)_\omega(p_1(\omega)) \in \mathcal{F} \times \mathcal{B}_X.$$

If $\mathcal{E}_1 = \emptyset$ then we stop. If, on the other hand, $\pi(\mathcal{E}_1) \in \mathcal{F}$ (using Lemma 4.2), and so again by Lemma 4.1 there exists a measurable map $p_2 : \pi(\mathcal{E}_1) \rightarrow X$ such that

$$e^{d_F(\omega, p_2(\omega))} \geq \sup_{x \in (\mathcal{E}_1)_\omega} e^{d_F(\omega, x)} - \frac{1}{2^{2+1}K} e^{-\|d_F(\omega)\|_\infty}$$

and $(\omega, p_2(\omega)) \in \mathcal{E}_1$ for each $\omega \in \pi(\mathcal{E}_1)$. Set

$$\mathcal{E}_2 = \mathcal{E}_1 \setminus \bigcup_{k=1}^K \bigcup_{\omega \in \pi(\mathcal{E}_1)} \{\omega\} \times ((\alpha_k)_F)_\omega(p_2(\omega)) \in \mathcal{F} \times \mathcal{B}_X.$$

It is not hard to see that, after finitely many steps of induction, we obtain $\mathcal{E}_0 \in \mathcal{F} \times \mathcal{B}_X$,

$$\mathcal{E}_j = \mathcal{E}_{j-1} \setminus \bigcup_{k=1}^K \bigcup_{\omega \in \pi(\mathcal{E}_{j-1})} \{\omega\} \times ((\alpha_k)_F)_\omega(p_j(\omega)) \in \mathcal{F} \times \mathcal{B}_X,$$

where $p_j : \pi(\mathcal{E}_{j-1}) \rightarrow X$ is a measurable map satisfying

$$e^{d_F(\omega, p_j(\omega))} \geq \sup_{x \in (\mathcal{E}_{j-1})_\omega} e^{d_F(\omega, x)} - \frac{1}{2^{j+1}K} e^{-\|d_F(\omega)\|_\infty}$$

and $(\omega, p_j(\omega)) \in \mathcal{E}_{j-1}$ for each $\omega \in \pi(\mathcal{E}_{j-1})$, $j = 1, \dots, m$ and $\mathcal{E}_{m-1} \neq \emptyset$, $\mathcal{E}_m = \emptyset$ (observe that, for $j = 1, \dots, m$ and $j_1, j_2 \in \{0, 1, \dots, j-1\}$, if $j_1 \neq j_2$ then $((\alpha_k)_F)_\omega(p_{j_1+1}(\omega))$ and $((\alpha_k)_F)_\omega(p_{j_2+1}(\omega))$ are different non-empty atoms of the partition $((\alpha_k)_F)_\omega$ for each $k = 1, \dots, K$ and any $\omega \in \pi(\mathcal{E}_{j-1})$, from this we could deduce that finally $\mathcal{E}_m = \emptyset$ after finite steps of induction).

Now for each $\omega \in \Omega$, set

$$B_{F,\omega} = \{p_j(\omega) : j \in \{1, \dots, m\}, \omega \in \mathcal{E}_{j-1}\}.$$

From the construction, it is easy to see that, for $\omega \in \Omega$, each atom of $((\alpha_k)_F)_\omega$ contains at most one point from $B_{F,\omega}$, $1 \leq k \leq K$ and $B_F \in \mathcal{F} \times \mathcal{B}_X$. To finish the proof, let $\omega \in \Omega$, we only need to check

$$\sum_{x \in B_{F,\omega}} e^{d_F(\omega, x)} > \frac{1}{K} \left[\inf_{\beta(\omega) \in \mathbf{P}_{\mathcal{E}_\omega, \beta(\omega)} \succeq (\mathcal{U}_F)_\omega} \sum_{B \in \beta(\omega)} \sup_{x \in B} e^{d_F(\omega, x)} - \frac{1}{2} e^{-\|d_F(\omega)\|_\infty} \right].$$

In fact, suppose that $m(\omega) \in \{1, \dots, m\}$ is the first $J \in \mathbb{N}$ such that $\omega \notin \pi(\mathcal{E}_J)$ and set

$$\gamma(\omega) = \{(\mathcal{E}_{j-1})_\omega \cap ((\alpha_k)_F)_\omega(p_j(\omega)) : j = 1, \dots, m(\omega), k = 1, \dots, K\}.$$

It is easy to check that $\gamma(\omega) \in \mathbf{C}_{\mathcal{E}_\omega}, \gamma(\omega) \succeq (\mathcal{U}_F)_\omega$. Moreover,

$$\begin{aligned}
& \sum_{x \in B_{F,\omega}} e^{d_F(\omega,x)} \\
&= \sum_{j=1}^{m(\omega)} e^{d_F(\omega,p_j(\omega))} \\
&\geq \sum_{j=1}^{m(\omega)} \frac{1}{K} \sum_{k=1}^K \left[\sup_{x \in (\mathcal{E}_{j-1})_\omega \cap ((\alpha_k)_F)_\omega(p_j(\omega))} e^{d_F(\omega,x)} - \frac{1}{2^{j+1}K} e^{-\|d_F(\omega)\|_\infty} \right] \\
&> \frac{1}{K} \left[\sum_{B(\omega) \in \gamma(\omega)} \sup_{x \in B(\omega)} e^{d_F(\omega,x)} - \frac{1}{2} e^{-\|d_F(\omega)\|_\infty} \right] \\
&\geq \frac{1}{K} \left[\inf_{\beta(\omega) \in \mathbf{P}_{\mathcal{E}_\omega}, \beta(\omega) \succeq (\mathcal{U}_F)_\omega} \sum_{B \in \beta(\omega)} \sup_{x \in B} e^{d_F(\omega,x)} - \frac{1}{2} e^{-\|d_F(\omega)\|_\infty} \right].
\end{aligned}$$

This finishes our proof. \square

Note that by the proof of [37, Lemma 3.1] we have (see also [20, Lemma 2.2], we should note that the assumption that (Y, \mathcal{D}, ν, G) is an MDS in [20, Lemma 2.2] is not necessary):

Lemma 8.4. *Let (Y, \mathcal{D}, ν) be a probability space, $\mathcal{C} \subseteq \mathcal{D}$ a sub- σ -algebra and $\alpha \in \mathbf{P}_Y$. Assume that G acts as a group of invertible measurable transformations (which may be not measure-preserving) over (Y, \mathcal{D}, ν) . If $E, F \in \mathcal{F}_G$ then*

$$H_\nu(\alpha_F|\mathcal{C}) \leq \sum_{g \in F} \frac{1}{|E|} H_\nu(\alpha_{Eg}|\mathcal{C}) + |F \setminus \{g \in G : E^{-1}g \subseteq F\}| \log |\alpha|.$$

The following result should be well known but we cannot find a reference for it, and so for completeness we present a proof of it here.

Lemma 8.5. *Let (Y, \mathcal{D}, ν_i) be a Lebesgue space, $i = 1, \dots, n, n \in \mathbb{N}$, $\mathcal{C} \subseteq \mathcal{D}$ a sub- σ -algebra and $0 < \lambda_1, \dots, \lambda_n < 1$ satisfy $\lambda_1 + \dots + \lambda_n = 1$. Then there exists $\lambda > 0$ (depending on $\lambda_1, \dots, \lambda_n$) such that, for each $\alpha \in \mathbf{P}_Y$,*

$$\lambda + \sum_{i=1}^n \lambda_i H_{\nu_i}(\alpha|\mathcal{C}) \geq H_{\lambda_1\nu_1 + \dots + \lambda_n\nu_n}(\alpha|\mathcal{C}) \geq \sum_{i=1}^n \lambda_i H_{\nu_i}(\alpha|\mathcal{C}).$$

Proof. We only consider the case of $n = 2$, as all the other cases follow from this case. By assumption, each (Y, \mathcal{C}, ν_i) is a Lebesgue space, $i = 1, 2$. Thus, there exists a sequence $\{\beta_i : i \in \mathbb{N}\} \subseteq \mathbf{P}_Y$ such that the sequence $\{\beta_i : i \in \mathbb{N}\}$ of σ -algebras increases to the σ -algebra \mathcal{C} in the sense of both ν_1 and ν_2 (and so also in the sense of $\lambda_1\nu_1 + \lambda_2\nu_2$), in particular,

$$(8.1) \quad \lim_{i \rightarrow \infty} H_\mu(\alpha|\beta_i) = H_\mu(\alpha|\mathcal{C})$$

whenever $\mu = \nu_1, \nu_2$ or $\lambda_1\nu_1 + \lambda_2\nu_2$. Now for each $i \in \mathbb{N}$, one has

$$(8.2) \quad \lambda_1 H_{\nu_1}(\alpha|\beta_i) + \lambda_2 H_{\nu_2}(\alpha|\beta_i) \leq H_{\lambda_1\nu_1 + \lambda_2\nu_2}(\alpha|\beta_i)$$

(by the proof of [35, Lemma 3.3 (1)]), and

$$\begin{aligned}
H_{\lambda_1\nu_1+\lambda_2\nu_2}(\alpha|\beta_i) &= H_{\lambda_1\nu_1+\lambda_2\nu_2}(\alpha \vee \beta_i) - H_{\lambda_1\nu_1+\lambda_2\nu_2}(\beta_i) \text{ (using (3.1))} \\
&\leq H_{\lambda_1\nu_1+\lambda_2\nu_2}(\alpha \vee \beta_i) - \sum_{j=1}^2 \lambda_j H_{\nu_j}(\beta_i) \text{ (using (8.2))} \\
&\leq \sum_{j=1}^2 \lambda_j H_{\nu_j}(\alpha \vee \beta_i) - \sum_{j=1}^2 \lambda_j \log \lambda_j - \sum_{j=1}^2 \lambda_j H_{\nu_j}(\beta_i) \\
&\quad \text{(by the proof of [67, Theorem 8.1])} \\
(8.3) \quad &= \sum_{j=1}^2 \lambda_j H_{\nu_j}(\alpha|\beta_i) - \sum_{j=1}^2 \lambda_j \log \lambda_j \text{ (using (3.1))}.
\end{aligned}$$

Combining (8.1) with (8.2) and (8.3) we obtain the required inequality. \square

Now we can prove:

Proposition 8.6. *Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a monotone sub-additive G -invariant family satisfying the assumption of (\spadesuit) and $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}^o$. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space and \mathcal{U} is good then, for some $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$,*

$$h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D}) \geq P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}).$$

Proof. As $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}^o$ is good, there exists a sequence $\{\alpha_n : n \in \mathbb{N}\} \subseteq \mathbf{P}_{\mathcal{U}}$ such that

- (a) for each $n \in \mathbb{N}$, $(\alpha_n)_{\omega}$ is a clopen partition of \mathcal{E}_{ω} for \mathbb{P} -a.e. $\omega \in \Omega$ and
- (b) $h_{\nu,+}^{(r)}(\mathbf{F}, \mathcal{U}) = \inf_{n \in \mathbb{N}} h_{\nu}^{(r)}(\mathbf{F}, \alpha_n)$ for each $\nu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$.

Observe that from our assumption of $e_G \subseteq F_1 \subsetneq F_2 \subsetneq \dots$ one has that $|F_n| \geq n$ for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ be fixed. By Lemma 8.3, there exists a family of finite subsets $B_{n,\omega} \subseteq \mathcal{E}_{\omega}, \omega \in \Omega$ such that

- (1) For $B_n \doteq \{(\omega, x) : \omega \in \Omega, x \in B_{n,\omega}\}$,

$$\sum_{x \in B_{n,\omega}} e^{d_{F_n}(\omega, x)} > \frac{1}{n} \left[\inf_{\beta(\omega) \in \mathbf{P}_{\mathcal{E}_{\omega}}, \beta(\omega) \succeq (\mathcal{U}_{F_n})_{\omega}} \sum_{B \in \beta(\omega)} \sup_{x \in B} e^{d_{F_n}(\omega, x)} - \frac{1}{2} e^{-\|d_{F_n}(\omega)\|_{\infty}} \right],$$

- (2) The family depends measurably on $\omega \in \Omega$ in the sense of $B_n \in \mathcal{F} \times \mathcal{B}_X$ and
- (3) Each atom of $((\alpha_k)_{F_n})_{\omega}$ contains at most one point from $B_{n,\omega}, 1 \leq k \leq n$.

Now we introduce a probability measure $\nu^{(n)}$ over \mathcal{E} by a measurable disintegration $d\nu^{(n)}(\omega, x) = d\nu_{\omega}^{(n)}(x)d\mathbb{P}(\omega)$, where

$$\nu_{\omega}^{(n)} = \sum_{x \in B_{n,\omega}} \frac{e^{d_{F_n}(\omega, x)} \delta_x}{\sum_{y \in B_{n,\omega}} e^{d_{F_n}(\omega, y)}},$$

and define another probability measure $\mu^{(n)}$ on \mathcal{E} by

$$\mu^{(n)} = \frac{1}{|F_n|} \sum_{g \in F_n} g\nu^{(n)}.$$

Observe that by assumption (2) the measure $\nu^{(n)}$ (and hence $\mu^{(n)}$) is well defined.

As the family \mathbf{D} satisfies (\spadesuit) , we can choose a sub-sequence $\{n_j : j \in \mathbb{N}\} \subseteq \mathbb{N}$ such that the sequence $\{\mu^{(n_j)} : j \in \mathbb{N}\}$ converges to some $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E})$ (and so $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$) and

$$(8.4) \quad \limsup_{j \rightarrow \infty} \frac{1}{|F_{n_j}|} \int_{\mathcal{E}} d_{F_{n_j}}(\omega, x) d\nu^{n_j}(\omega, x) \leq \mu(\mathbf{D}).$$

To finish our proof, by the selection of the sequence $\{\alpha_n : n \in \mathbb{N}\}$ it suffices to prove $h_{\mu}^{(r)}(\mathbf{F}, \alpha_l) + \mu(\mathbf{D}) \geq P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F})$ for each $l \in \mathbb{N}$. Let $l \in \mathbb{N}$ be fixed.

For each $n > l$, from the construction of $\nu_{\omega}^{(n)}$, one has

$$(8.5) \quad \begin{aligned} H_{\nu_{\omega}^{(n)}}(((\alpha_l)_{F_n})_{\omega}) &= \sum_{x \in B_{n, \omega}} - \frac{e^{d_{F_n}(\omega, x)}}{\sum_{y \in B_{n, \omega}} e^{d_{F_n}(\omega, y)}} \log \frac{e^{d_{F_n}(\omega, x)}}{\sum_{y \in B_{n, \omega}} e^{d_{F_n}(\omega, y)}} \\ &= \sum_{x \in B_{n, \omega}} - \frac{e^{d_{F_n}(\omega, x)} d_{F_n}(\omega, x)}{\sum_{y \in B_{n, \omega}} e^{d_{F_n}(\omega, y)}} + \log \sum_{y \in B_{n, \omega}} e^{d_{F_n}(\omega, y)} \\ &= - \int_X d_{F_n}(\omega, x) d\nu_{\omega}^{(n)}(x) + \log \sum_{y \in B_{n, \omega}} e^{d_{F_n}(\omega, y)}, \end{aligned}$$

as each atom of $((\alpha_l)_{F_n})_{\omega}$ contains at most one point from $B_{n, \omega}$. This implies

$$(8.6) \quad \begin{aligned} &\log P_{\mathcal{E}}(\omega, \mathbf{D}, F_n, \mathcal{U}, \mathbf{F}) - \log 2 - \log n \\ &\leq \log \left[P_{\mathcal{E}}(\omega, \mathbf{D}, F_n, \mathcal{U}, \mathbf{F}) - \frac{1}{2} e^{-\|d_{F_n}(\omega)\|_{\infty}} \right] - \log n \text{ (from the definitions)} \\ &= \log \left[\inf_{\beta(\omega) \in \mathbf{P}_{\mathcal{E}, \omega}, \beta(\omega) \succeq (\mathcal{U}_{F_n})_{\omega}} \sum_{B \in \beta(\omega)} \sup_{x \in B} e^{d_{F_n}(\omega, x)} - \frac{1}{2} e^{-\|d_{F_n}(\omega)\|_{\infty}} \right] - \log n \\ &< \log \sum_{x \in B_{n, \omega}} e^{d_{F_n}(\omega, x)} \text{ (by the assumption of (1))} \\ &= H_{\nu_{\omega}^{(n)}}(((\alpha_l)_{F_n})_{\omega}) + \int_X d_{F_n}(\omega, x) d\nu_{\omega}^{(n)}(x) \text{ (using (8.5))}, \end{aligned}$$

and so by Proposition 5.4 (1) and the construction of $\nu^{(n)}$ (using (4.4)), for each $B \in \mathcal{F}_G$ we have

$$\begin{aligned}
& \int_{\Omega} \log P_{\mathcal{E}}(\omega, \mathbf{D}, F_n, \mathcal{U}, \mathbf{F}) d\mathbb{P}(\omega) - \log 2 - \log n \\
& < H_{\nu^{(n)}}((\alpha_l)_{F_n} | \mathcal{F}_{\mathcal{E}}) + \int_{\mathcal{E}} d_{F_n}(\omega, x) d\nu^{(n)}(\omega, x) \\
& \leq \sum_{g \in F_n} \frac{1}{|B|} H_{\nu^{(n)}}((\alpha_l)_{Bg} | \mathcal{F}_{\mathcal{E}}) + |F_n \setminus \{g \in G : B^{-1}g \subseteq F_n\}| \log |\alpha_l| \\
& \quad + \int_{\mathcal{E}} d_{F_n}(\omega, x) d\nu^{(n)}(\omega, x) \text{ (using Lemma 8.4)} \\
& = \frac{|F_n|}{|B|} \sum_{g \in F_n} \frac{1}{|F_n|} H_{g\nu^{(n)}}((\alpha_l)_B | \mathcal{F}_{\mathcal{E}}) + |F_n \setminus \{g \in G : B^{-1}g \subseteq F_n\}| \log |\alpha_l| \\
& \quad + \int_{\mathcal{E}} d_{F_n}(\omega, x) d\nu^{(n)}(\omega, x) \text{ (using the } G\text{-invariance of } \mathcal{F}_{\mathcal{E}}) \\
& \leq \frac{|F_n|}{|B|} H_{\mu^{(n)}}((\alpha_l)_B | \mathcal{F}_{\mathcal{E}}) + |F_n \setminus \{g \in G : B^{-1}g \subseteq F_n\}| \log |\alpha_l| \\
(8.7) \quad & + \int_{\mathcal{E}} d_{F_n}(\omega, x) d\nu^{(n)}(\omega, x) \text{ (using Lemma 8.5).}
\end{aligned}$$

Let $B \in \mathcal{F}_G$ be fixed. Observe that, as $\{F_n : n \in \mathbb{N}\}$ is a Følner sequence,

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} |F_n \setminus \{g \in G : B^{-1}g \subseteq F_n\}| = 0;$$

moreover, by the selection of α_l , one has that $((\alpha_l)_B)_{\omega}$ is a clopen partition of \mathcal{E}_{ω} for \mathbb{P} -a.e. $\omega \in \Omega$, and so we have (using Proposition 4.5 (2))

$$\limsup_{n \rightarrow \infty} H_{\mu^{(n)}}((\alpha_l)_B | \mathcal{F}_{\mathcal{E}}) \leq H_{\mu}((\alpha_l)_B | \mathcal{F}_{\mathcal{E}}).$$

Combined with (8.7) (divided by $|F_n|$, recall $|F_n| \geq n$) we obtain (using (8.4))

$$P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) \leq \frac{1}{|B|} H_{\mu}((\alpha_l)_B | \mathcal{F}_{\mathcal{E}}) + \mu(\mathbf{D}).$$

Lastly, taking the infimum over all $B \in \mathcal{F}_G$ we obtain

$$P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) \leq h_{\mu}(G, \alpha_l | \mathcal{F}_{\mathcal{E}}) + \mu(\mathbf{D}),$$

equivalently, $P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) \leq h_{\mu}^{(r)}(\mathbf{F}, \alpha_l) + \mu(\mathbf{D})$. This ends the proof. \square

Now we can present the proof of Proposition 8.1.

Proof of Proposition 8.1. As \mathcal{U} is factor good, then there exists a family $\mathbf{F}' = \{F'_{g,\omega} : \mathcal{E}'_{\omega} \rightarrow \mathcal{E}'_{g\omega} | g \in G, \omega \in \Omega\}$ (with a compact metric state space X' and $\mathcal{E}' \in \mathcal{F} \times \mathcal{B}_{X'}$) which is a continuous bundle RDS over $(\Omega, \mathcal{F}, \mathbb{P}, G)$ and factor map $\pi : \mathcal{E}' \rightarrow \mathcal{E}$ such that $\pi^{-1}\mathcal{U}$ is good. By Lemma 6.8 and Lemma 7.8, $\mathbf{D} \circ \pi$ is a monotone sub-additive G -invariant family satisfying (\spadesuit) , and so there exists $\mu' \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}', G)$ such that (using Proposition 8.6)

$$h_{\mu',+}^{(r)}(\mathbf{F}', \pi^{-1}\mathcal{U}) + \mu'(\mathbf{D} \circ \pi) \geq P_{\mathcal{E}'}(\mathbf{D} \circ \pi, \pi^{-1}\mathcal{U}, \mathbf{F}').$$

Set $\mu = \pi\mu'$. Observe that, using Lemma 6.8, we have $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$,

$$h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) \geq h_{\mu',+}^{(r)}(\mathbf{F}', \pi^{-1}\mathcal{U})$$

and

$$P_{\mathcal{E}'}(\mathbf{D} \circ \pi, \pi^{-1}\mathcal{U}, \mathbf{F}') = P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}).$$

From the definition one sees easily that $\mu'(\mathbf{D} \circ \pi) = \mu(\mathbf{D})$ and hence

$$h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D}) \geq P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}).$$

This finishes our proof. \square

9. ASSUMPTION (\spadesuit) ON THE FAMILY \mathbf{D}

In this section, we discuss (\spadesuit) on the family \mathbf{D} .

Before proceeding, we need introduce the property of strong sub-additivity. In his treatment of entropy theory for amenable group actions Moullin-Ollagnier [56] used this property rather heavily.

Let (Y, \mathcal{D}, ν, G) be an MDS and $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq L^1(Y, \mathcal{D}, \nu)$. \mathbf{D} is called *strongly sub-additive* if for ν -a.e. $y \in Y$,

$$d_{E \cup F}(y) + d_{E \cap F}(y) \leq d_E(y) + d_F(y)$$

whenever $E, F \in \mathcal{F}_G$ (here we set $d_\emptyset(y) = 0$ for ν -a.e. $y \in Y$ by convention). For an invariant family, the property of strong sub-additivity is stronger than the property of sub-additivity, and \mathbf{D}^f is a strongly sub-additive G -invariant family in $L^1(Y, \mathcal{D}, \nu)$ for each $f \in L^1(Y, \mathcal{D}, \nu)$. Similarly, we can introduce strong sub-additivity for any given continuous bundle RDS.

Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a strongly sub-additive G -invariant family. By Proposition 2.3, for each $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ we may still define

$$\begin{aligned} \mu(\mathbf{D}) &= \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\mathcal{E}} d_{F_n}(\omega, x) d\mu(\omega, x) \\ (9.1) \quad &= \inf_{n \in \mathbb{N}} \frac{1}{|F_n|} \int_{\mathcal{E}} d_{F_n}(\omega, x) d\mu(\omega, x). \end{aligned}$$

Remark that the value of $\mu(\mathbf{D})$ is independent of the choice of Følner sequence $\{F_n : n \in \mathbb{N}\}$. The points of difference from the case where \mathbb{D} is a monotone sub-additive G -invariant family are:

- (1) $\mu(\mathbf{D})$ need not to be non-negative, in fact, it may take the value $-\infty$, as here \mathbf{D} may be not monotone. Thus \mathbf{D} need not to be non-negative. A direct example is to set $d_F(\omega, x)$ to be the constant function $-|F|^2$ for each $F \in \mathcal{F}_G$.
- (2) By (9.1), the function

$$\bullet(\mathbf{D}) : \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G) \rightarrow \mathbb{R} \cup \{-\infty\}, \mu \mapsto \mu(\mathbf{D}),$$

is the infimum of a family of continuous functions, and hence is u.s.c.

- (3) Observe that the family

$$\left\{ \sup_{x \in \mathcal{E}_\omega} d_F(\omega, x) : F \in \mathcal{F}_G \right\} \subseteq L^1(\Omega, \mathcal{F}, \mathbb{P})$$

may be not strongly sub-additive, as for $E, F \in \mathcal{F}_G$ it may happen

$$\sup_{x \in \mathcal{E}_\omega} d_{E \cap F}(\omega, x) + \sup_{x \in \mathcal{E}_\omega} d_{E \cup F}(\omega, x) > \sup_{x \in \mathcal{E}_\omega} d_E(\omega, x) + \sup_{x \in \mathcal{E}_\omega} d_F(\omega, x)$$

even if

$$d_{E \cap F}(\omega, x) + d_{E \cup F}(\omega, x) \leq d_E(\omega, x) + d_F(\omega, x).$$

Thus we can not define $\sup_{\mathbb{P}}(\mathbf{D})$ similarly.

Remark 9.1. Let $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}$. In this setting, it may happen that the family $\{\log P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}) : F \in \mathcal{F}_G\}$ is not strongly sub-additive.

The following result tells that we can remove the assumption of (\spadesuit) if we require the additional property of strong sub-additivity over the family: note that it is not necessary to assume the family monotone.

Proposition 9.2. Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a strongly sub-additive G -invariant family. Then \mathbf{D} satisfies (\spadesuit) .

In order to prove Proposition 9.2, we need the following lemma.

Lemma 9.3. Let (Y, \mathcal{D}, ν, G) be an MDS and $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\}$ a strongly sub-additive family in $L^1(Y, \mathcal{D}, \nu)$. If $E, E_1, \dots, E_n \in \mathcal{F}_G, n \in \mathbb{N}$ satisfy

$$1_E = \sum_{i=1}^n a_i 1_{E_i},$$

where all $a_1, \dots, a_n > 0$ are rational numbers, then

$$d_E(y) \leq \sum_{i=1}^n a_i d_{E_i}(y)$$

for ν -a.e. $y \in Y$. A similar result holds for a continuous bundle RDS.

Proof. First, we consider the case of $a_1 = \dots = a_n = \frac{1}{m}$ for some $m \in \mathbb{N}$. Obviously $\bigcup_{i=1}^n E_i = E$. Say (neglecting all empty elements)

$$\{A_1, \dots, A_p\} = \bigvee_{i=1}^n \{E_i, E \setminus E_i\}.$$

Set $K_0 = \emptyset$, $K_i = \bigcup_{j=1}^i A_j$, $i = 1, \dots, p$. Then $\emptyset = K_0 \subsetneq K_1 \subsetneq \dots \subsetneq K_p = E$.

Moreover, if for some $i = 1, \dots, p$ and $j = 1, \dots, n$ with $E_j \cap (K_i \setminus K_{i-1}) \neq \emptyset$ then $K_i \setminus K_{i-1} \subseteq E_j$ and so $K_i = K_{i-1} \cup (K_i \cap E_j)$, thus, for ν -a.e. $y \in Y$,

$$d_{K_i}(y) + d_{K_{i-1} \cap E_j}(y) \leq d_{K_{i-1}}(y) + d_{K_i \cap E_j}(y),$$

i.e.

$$(9.2) \quad d_{K_i}(y) - d_{K_{i-1}}(y) \leq d_{K_i \cap E_j}(y) - d_{K_{i-1} \cap E_j}(y),$$

as the family \mathbf{D} is strongly sub-additive. Now for each $i = 1, \dots, p$ we can select $k_i \in K_i \setminus K_{i-1}$, observe that if $k_i \notin E_j$ then $E_j \cap (K_i \setminus K_{i-1}) = \emptyset$ (and hence

$K_i \cap E_j = K_{i-1} \cap E_j$) for $j = 1, \dots, n$, and so one has that, for ν -a.e. $y \in Y$,

$$\begin{aligned}
d_E(y) &= \sum_{i=1}^p (d_{K_i}(y) - d_{K_{i-1}}(y)) \text{ (by the construction of } K_0, K_1, \dots, K_p) \\
&= \sum_{i=1}^p \left(\frac{1}{m} \sum_{j=1}^n 1_{E_j}(k_i) \right) (d_{K_i}(y) - d_{K_{i-1}}(y)) \text{ (by assumptions)} \\
&= \frac{1}{m} \sum_{j=1}^n \sum_{1 \leq i \leq p, k_i \in E_j} (d_{K_i}(y) - d_{K_{i-1}}(y)) \\
&\leq \frac{1}{m} \sum_{j=1}^n \sum_{1 \leq i \leq p, k_i \in E_j} (d_{K_i \cap E_j}(y) - d_{K_{i-1} \cap E_j}(y)) \text{ (using (9.2))} \\
&= \frac{1}{m} \sum_{j=1}^n \sum_{i=1}^p (d_{K_i \cap E_j}(y) - d_{K_{i-1} \cap E_j}(y)) \\
&\quad \text{(as if } k_i \notin E_j \text{ then } K_i \cap E_j = K_{i-1} \cap E_j) \\
&= \frac{1}{m} \sum_{j=1}^n d_{E_j}(y).
\end{aligned}$$

The general case follows easily from the above special case. \square

Proof of Proposition 9.2. The proof is partly inspired by that of Proposition 2.3.

Let $\{\nu_n : n \in \mathbb{N}\} \subseteq \mathcal{P}_{\mathbb{P}}(\mathcal{E})$ be a given sequence. Set $\mu_n = \frac{1}{|F_n|} \sum_{g \in F_n} g\nu_n$ for each $n \in \mathbb{N}$. By Proposition 4.5 there exists a sub-sequence $\{n_j : j \in \mathbb{N}\} \subseteq \mathbb{N}$ such that the sequence $\{\mu_{n_j} : j \in \mathbb{N}\}$ converges to some $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$. Now we check

$$(9.3) \quad \limsup_{j \rightarrow \infty} \frac{1}{|F_{n_j}|} \int_{\mathcal{E}} d_{F_{n_j}}(\omega, x) d\nu_{n_j}(\omega, x) \leq \mu(\mathbf{D}).$$

For each $F \in \mathcal{F}_G$ set

$$d'_F(\omega, x) = d_F(\omega, x) - \sum_{g \in F} d_{\{e_G\}}(g(\omega, x))$$

and put

$$\mathbf{D}' = \{d'_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X)).$$

As \mathbf{D} is a strongly sub-additive G -invariant family, then the family \mathbf{D}' is also strongly sub-additive G -invariant and $-\mathbf{D}'$ is non-negative. Observe that

$$\begin{aligned}
&\limsup_{j \rightarrow \infty} \frac{1}{|F_{n_j}|} \int_{\mathcal{E}} d_{F_{n_j}}(\omega, x) d\nu_{n_j}(\omega, x) \\
&= \limsup_{j \rightarrow \infty} \frac{1}{|F_{n_j}|} \int_{\mathcal{E}} d'_{F_{n_j}}(\omega, x) d\nu_{n_j}(\omega, x) + \limsup_{j \rightarrow \infty} \int_{\mathcal{E}} d_{\{e_G\}}(\omega, x) d\mu_{n_j}(\omega, x) \\
(9.4) \quad &= \limsup_{j \rightarrow \infty} \frac{1}{|F_{n_j}|} \int_{\mathcal{E}} d'_{F_{n_j}}(\omega, x) d\nu_{n_j}(\omega, x) + \int_{\mathcal{E}} d_{\{e_G\}}(\omega, x) d\mu(\omega, x) \\
&\quad \text{(as the sequence } \{\mu_{n_j} : j \in \mathbb{N}\} \text{ converges to } \mu)
\end{aligned}$$

and

$$\begin{aligned}
\mu(\mathbf{D}) &= \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\mathcal{E}} d_{F_n}(\omega, x) d\mu(\omega, x) \\
&= \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\mathcal{E}} d'_{F_n}(\omega, x) d\mu(\omega, x) \\
&\quad + \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\mathcal{E}} \sum_{g \in F_n} d_{\{e_G\}}(g(\omega, x)) d\mu(\omega, x) \\
(9.5) \quad &= \mu(\mathbf{D}') + \int_{\mathcal{E}} d_{\{e_G\}}(\omega, x) d\mu(\omega, x) \text{ (as } \mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)).
\end{aligned}$$

To prove (9.3), by (9.4) and (9.5), we only need prove

$$(9.6) \quad \limsup_{j \rightarrow \infty} \frac{1}{|F_{n_j}|} \int_{\mathcal{E}} d'_{F_{n_j}}(\omega, x) d\nu_{n_j}(\omega, x) \leq \mu(\mathbf{D}').$$

Let $T \in \mathcal{F}_G$ be fixed. As $\{F_n : n \in \mathbb{N}\}$ is a Følner sequence of G , for each $n \in \mathbb{N}$ we set $E_n = F_n \cap \bigcap_{g \in T} g^{-1}F_n \subseteq F_n$, then $\lim_{n \rightarrow \infty} \frac{|E_n|}{|F_n|} = 1$. Set

$$w_n = \frac{1}{|E_n|} \sum_{g \in E_n} g\nu_n \text{ for each } n \in \mathbb{N}.$$

Observe that the sequence $\{\mu_{n_j} : j \in \mathbb{N}\}$ converges to μ . By the selection of $E_n, n \in \mathbb{N}$, it is easy to see that the sequence $\{w_{n_j} : j \in \mathbb{N}\}$ also converges to μ .

Now for each $n \in \mathbb{N}$, using Lemma 2.5, one has

$$\sum_{t \in T} 1_{tE_n} = \sum_{g \in E_n} 1_{Tg}.$$

By the construction of E_n , $tE_n \subseteq F_n$ for any $t \in T$, there exist $E'_1, \dots, E'_m \in \mathcal{F}_G, m \in \{0\} \cup \mathbb{N}$ and rational numbers $a_1, \dots, a_m > 0$ such that

$$1_{F_n} = \frac{1}{|T|} \sum_{t \in T} 1_{tE_n} + \sum_{j=1}^m a_j 1_{E'_j},$$

and so

$$(9.7) \quad 1_{F_n} = \frac{1}{|T|} \sum_{g \in E_n} 1_{Tg} + \sum_{j=1}^m a_j 1_{E'_j},$$

which implies that, for \mathbb{P} -a.e. $\omega \in \Omega$,

$$\begin{aligned}
d'_{F_n}(\omega, x) &\leq \frac{1}{|T|} \sum_{g \in E_n} d'_{Tg}(\omega, x) + \sum_{j=1}^m a_j d'_{E'_j}(\omega, x) \\
&\quad \text{(using Lemma 9.3, as the family } \mathbf{D}' \text{ is strongly sub-additive)} \\
&\leq \frac{1}{|T|} \sum_{g \in E_n} d'_{Tg}(\omega, x) \text{ (as the family } -\mathbf{D}' \text{ is non-negative)} \\
(9.8) \quad &= \frac{1}{|T|} \sum_{g \in E_n} d'_T(g(\omega, x)) \text{ (as the family } \mathbf{D}' \text{ is } G\text{-invariant)}
\end{aligned}$$

for each $x \in \mathcal{E}_\omega$, and so

$$\begin{aligned}
& \limsup_{j \rightarrow \infty} \frac{1}{|F_{n_j}|} \int_{\mathcal{E}} d'_{F_{n_j}}(\omega, x) d\nu_{n_j}(\omega, x) \\
&= \limsup_{j \rightarrow \infty} \frac{1}{|E_{n_j}|} \int_{\mathcal{E}} d'_{F_{n_j}}(\omega, x) d\nu_{n_j}(\omega, x) \text{ (by the selection of } E_{n_j}) \\
&\leq \limsup_{j \rightarrow \infty} \frac{1}{|T|} \int_{\mathcal{E}} d'_T(\omega, x) dw_{n_j}(\omega, x) \text{ (using (9.8))} \\
&= \frac{1}{|T|} \int_{\mathcal{E}} d'_T(\omega, x) d\mu(\omega, x) \text{ (as the sequence } \{w_{n_j} : j \in \mathbb{N}\} \text{ converges to } \mu).
\end{aligned}$$

which implies (9.6) (combined with (9.1)). This finishes our proof. \square

10. THE LOCAL VARIATIONAL PRINCIPLE IN SOME SPECIAL CASES

In this section we aim to discuss the local variational principle for fiber topological pressure in the case of amenable groups admitting a tiling Følner sequence. Thus, throughout this section, we assume that each $F_n, n \in \mathbb{N}$ is a subset tiling G .

Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a sub-additive G -invariant family and $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}$. Then by Proposition 2.8 and Proposition 5.4 we may introduce

$$\begin{aligned}
P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) &= \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\Omega} \log P_{\mathcal{E}}(\omega, \mathbf{D}, F_n, \mathcal{U}, \mathbf{F}) d\mathbb{P}(\omega) \\
&= \inf_{n \in \mathbb{N}} \frac{1}{|F_n|} \int_{\Omega} \log P_{\mathcal{E}}(\omega, \mathbf{D}, F_n, \mathcal{U}, \mathbf{F}) d\mathbb{P}(\omega)
\end{aligned}$$

and

$$P_{\mathcal{E}}(\mathbf{D}, \mathbf{F}) = \sup_{\mathcal{V} \in \mathbf{C}_{\mathcal{E}}^{\circ}} P_{\mathcal{E}}(\mathbf{D}, (\Omega \times \mathcal{V})_{\mathcal{E}}, \mathbf{F}),$$

which we still call the *fiber topological \mathbf{D} -pressure of \mathbf{F} with respect to \mathcal{U}* and the *fiber topological \mathbf{D} -pressure of \mathbf{F}* , respectively. By the same reasoning, for each $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ we can define

$$\begin{aligned}
\mu(\mathbf{D}) &= \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\mathcal{E}} d_{F_n}(\omega, x) d\mu(\omega, x) \\
(10.1) \quad &= \inf_{n \in \mathbb{N}} \frac{1}{|F_n|} \int_{\mathcal{E}} d_{F_n}(\omega, x) d\mu(\omega, x)
\end{aligned}$$

and

$$\sup_{\mathbb{P}}(\mathbf{D}) = \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\Omega} \sup_{x \in \mathcal{E}_\omega} d_F(\omega, x) d\mathbb{P}(\omega) \geq \mu(\mathbf{D}).$$

As above, all these invariants are independent of the selection of the Følner sequence $\{F_n : n \in \mathbb{N}\}$. Moreover, as in §9, neither $\mu(\mathbf{D})$ nor $\sup_{\mathbb{P}}(\mathbf{D})$ need be non-negative (in fact, they may take the value of $-\infty$), and the function $\bullet(\mathbf{D}) : \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G) \rightarrow \mathbb{R} \cup \{-\infty\}, \mu \mapsto \mu(\mathbf{D})$ is u.s.c.

Almost all the definitions and theorems in the previous sections can be carried out unchanged in our present setting. We skip most of them, and emphasize only some of them as follows.

As in Proposition 5.6 and Proposition 5.8 one has:

Proposition 10.1. *Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a sub-additive G -invariant family and $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}, \mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$. Then*

- (1) $P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) \geq h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D})$ and if $\mu(\mathbf{D}) > -\infty$ then $P_{\mathcal{E}}(\mathbf{D}, \mathbf{F}) \geq h_{\mu}^{(r)}(\mathbf{F}) + \mu(\mathbf{D})$.
- (2) $\sup_{\mathbb{P}}(\mathbf{D}) \leq P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) \leq h_{\text{top}}^{(r)}(\mathbf{F}, \mathcal{U}) + \sup_{\mathbb{P}}(\mathbf{D})$ and if $\sup_{\mathbb{P}}(\mathbf{D}) = -\infty$ then $P_{\mathcal{E}}(\mathbf{D}, \mathbf{F}) = -\infty$.

Proof. The proof is similar to that of Proposition 5.6 and Proposition 5.8, except that if $\sup_{\mathbb{P}}(\mathbf{D}) = -\infty$ then $P_{\mathcal{E}}(\mathbf{D}, \mathbf{F}) = -\infty$. In fact, if $\sup_{\mathbb{P}}(\mathbf{D}) = -\infty$ then by the inequality

$$\sup_{\mathbb{P}}(\mathbf{D}) \leq P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) \leq h_{\text{top}}^{(r)}(\mathbf{F}, \mathcal{U}) + \sup_{\mathbb{P}}(\mathbf{D})$$

one has $P_{\mathcal{E}}(\mathbf{D}, \mathcal{V}, \mathbf{F}) = -\infty$ for each $\mathcal{V} \in \mathbf{C}_{\mathcal{E}}$, which implies $P_{\mathcal{E}}(\mathbf{D}, \mathbf{F}) = -\infty$. \square

Moreover, as for our main results Theorem 7.1 and Proposition 7.7, we have:

Theorem 10.2. *Let $\mathcal{U} \in \mathbf{C}_{\mathcal{E}}^{\circ}$. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space and \mathcal{U} is factor good.*

- (1) *If $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ is a sub-additive G -invariant family satisfying the assumption of (\spadesuit) then*

$$P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) = \max_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D})] = \max_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D})],$$

$$\sup_{\mathbb{P}}(\mathbf{D}) = \max_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} \mu(\mathbf{D}).$$

- (2) *If $f \in \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ then*

$$\begin{aligned} P_{\mathcal{E}}(\mathbf{D}^f, \mathcal{U}, \mathbf{F}) &= \max_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) + \int_{\mathcal{E}} f(\omega, x) d\mu(\omega, x)] \\ &= \max_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}) + \int_{\mathcal{E}} f(\omega, x) d\mu(\omega, x)]. \end{aligned}$$

Proof. (1) The proof is just a re-writing of Theorem 7.1 and Proposition 7.7 (see also Remark 7.3).

(2) This is just a special case of (1). In fact, if $f \in \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ then obviously \mathbf{D}^f is a sub-additive G -invariant family satisfying (\spadesuit) and $\mu(\mathbf{D}^f) = \int_{\mathcal{E}} f(\omega, x) d\mu(\omega, x)$. Thus, the conclusion follows from (1). \square

Combined with Theorem 4.11 and Proposition 10.1, a direct corollary of Theorem 10.2, is (see [11, 74] and [75, Theorem 4.1] for the special case of $G = \mathbb{Z}$):

Corollary 10.3. *Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ is a sub-additive G -invariant family satisfying the assumption of (\spadesuit) . Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space. Then*

$$P_{\mathcal{E}}(\mathbf{D}, \mathbf{F}) = \begin{cases} -\infty, & \text{if } \sup_{\mathbb{P}}(\mathbf{D}) = -\infty \\ \sup_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G), \mu(\mathbf{D}) > -\infty} [h_{\mu}^{(r)}(\mathbf{F}) + \mu(\mathbf{D})], & \text{otherwise} \end{cases}.$$

In particular, for each $f \in \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ one has

$$P_{\mathcal{E}}(\mathbf{D}^f, \mathbf{F}) = \sup_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}) + \int_{\mathcal{E}} f(\omega, x) d\mu(\omega, x)].$$

Excepting the discussions in §9, given a sub-additive G -invariant family \mathbf{D} , it may not be easy to check whether \mathbf{D} satisfies (\spadesuit) .

In the remainder of this section, we will discuss the special case where the group G is abelian, which shows us that to some extent this assumption is quite natural.

Proposition 10.4. *Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a sub-additive G -invariant family. If G is abelian then \mathbf{D} satisfies the assumption of (\spadesuit) .*

Remark 10.5. *Observe that the special case of $G = \mathbb{Z}$ in the absolute setting was first obtained by Cao, Feng and Huang [11, Lemma 2.3], and so [11, Theorem 1.1] (see also [74, Theorem 6.4] and its local version [74, Theorem 4.5]) follows from Corollary 10.3 (and its local version Theorem 10.2).*

Before proving Proposition 10.4, we make the following observation.

Lemma 10.6. *Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a sub-additive G -invariant family and $T \in \mathcal{T}_G, \epsilon > 0$. Assume that G is abelian and the family $-\mathbf{D}$ is non-negative. Then, whenever $n \in \mathbb{N}$ is sufficiently large, there exists $H_n \subseteq F_n$ such that $|F_n \setminus H_n| \leq 2\epsilon|F_n|$ and, for \mathbb{P} -a.e. $\omega \in \Omega$,*

$$d_{F_n}(\omega, x) \leq \frac{1}{|T|} \sum_{g \in H_n} d_T(g(\omega, x)) \text{ for each } x \in \mathcal{E}_{\omega}.$$

Proof. As $T \in \mathcal{T}_G$ and $\{F_n : n \in \mathbb{N}\}$ is a Følner sequence of G , if only $n \in \mathbb{N}$ is sufficiently large then there exists $E_n \in \mathcal{F}_G$ such that $Tg, g \in E_n$ are pairwise disjoint, $TE_n \subseteq T_n \doteq F_n \cap \bigcap_{t \in T} t^{-1}F_n$ and $|TE_n| \geq |T_n| - \epsilon|F_n|, |T_n| \geq (1 - \epsilon)|F_n|$ (hence $|TE_n| \geq (1 - 2\epsilon)|F_n|$). As \mathbf{D} is a sub-additive G -invariant family, $-\mathbf{D}$ is non-negative and G is abelian, then, for \mathbb{P} -a.e. $\omega \in \Omega$,

$$\begin{aligned} d_{F_n}(\omega, x) &\leq d_{tT_n}(\omega, x) + d_{F_n \setminus tT_n}(\omega, x) \text{ (as } tT_n \subseteq F_n) \\ &\leq d_{tTE_n}(\omega, x) + d_{t(T_n \setminus TE_n)}(\omega, x) \text{ (as } TE_n \subseteq T_n) \\ &\leq \sum_{g \in E_n} d_{tT}(g(\omega, x)) \text{ (as } Tg, g \in E_n \text{ are pairwise disjoint)} \\ (10.2) \quad &= \sum_{g \in E_n} d_{Tt}(g(\omega, x)) = \sum_{g \in E_n} d_T(tg(\omega, x)) \end{aligned}$$

for each $t \in T$ and any $x \in \mathcal{E}_{\omega}$. Summing up (10.2) over all $t \in T$ we obtain:

$$(10.3) \quad |T|d_{F_n}(\omega, x) \leq \sum_{g \in TE_n} d_T(g(\omega, x))$$

for \mathbb{P} -a.e. $\omega \in \Omega$ and each $x \in \mathcal{E}_{\omega}$ (observe that $Tg, g \in E_n$ are pairwise disjoint). The theorem follows by setting $H_n = TE_n$. \square

Now let us finish the proof of Proposition 10.4.

Proof of Proposition 10.4. The proof is based on that of Proposition 2.8.

Let $\{\nu_n : n \in \mathbb{N}\} \subseteq \mathcal{P}_{\mathbb{P}}(\mathcal{E})$ be a given sequence. Set

$$\mu_n = \frac{1}{|F_n|} \sum_{g \in F_n} g\nu_n \text{ for each } n \in \mathbb{N}.$$

By Proposition 4.5 there exists some sub-sequence $\{n_j : j \in \mathbb{N}\} \subseteq \mathbb{N}$ such that the sequence $\{\mu_{n_j} : j \in \mathbb{N}\}$ converges to some $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$. Now we aim to check

$$(10.4) \quad \limsup_{j \rightarrow \infty} \frac{1}{|F_{n_j}|} \int_{\mathcal{E}} d_{F_{n_j}}(\omega, x) d\nu_{n_j}(\omega, x) \leq \mu(\mathbf{D}).$$

As in the proof of Proposition 9.2, we may assume that the family $-\mathbb{D}$ is non-negative. Applying Lemma 10.6 to \mathbf{D} we see that, if we fix $T \in \mathcal{T}_G$ and $\epsilon > 0$, and if $n \in \mathbb{N}$ is sufficiently large then there exists $T_n \subseteq F_n$ such that $|F_n \setminus T_n| \leq 2\epsilon|F_n|$ and, for \mathbb{P} -a.e. $\omega \in \Omega$,

$$(10.5) \quad d_{F_n}(\omega, x) \leq \frac{1}{|T|} \sum_{g \in T_n} d_T(g(\omega, x)) \text{ for each } x \in \mathcal{E}_\omega.$$

We see from this that we may assume without loss of generality that $T_n \subseteq F_n$ satisfies $\lim_{n \rightarrow \infty} \frac{|T_n|}{|F_n|} = 1$ and, for \mathbb{P} -a.e. $\omega \in \Omega$, (10.5) holds for sufficiently large $n \in \mathbb{N}$. Set

$$w_n = \frac{1}{|T_n|} \sum_{g \in T_n} g\nu_n \text{ for each large enough } n \in \mathbb{N}.$$

Observe that the sequence $\{\mu_{n_j} : j \in \mathbb{N}\}$ converges to μ . By the choice of $T_n, n \in \mathbb{N}$, it is easy to see that the sequence $\{w_{n_j} : j \in \mathbb{N}\}$ also converges to μ . Thus

$$\begin{aligned} & \limsup_{j \rightarrow \infty} \frac{1}{|F_{n_j}|} \int_{\mathcal{E}} d_{F_{n_j}}(\omega, x) d\nu_{n_j}(\omega, x) \\ & \leq \limsup_{j \rightarrow \infty} \frac{1}{|F_{n_j}|} \int_{\mathcal{E}} \frac{1}{|T|} \sum_{g \in T_{n_j}} d_T(g(\omega, x)) d\nu_{n_j}(\omega, x) \text{ (using (10.5))} \\ & = \limsup_{j \rightarrow \infty} \frac{1}{|T_{n_j}|} \int_{\mathcal{E}} \frac{1}{|T|} \sum_{g \in T_{n_j}} d_T(g(\omega, x)) d\nu_{n_j}(\omega, x) \text{ (by the selection of } T_{n_j}) \\ & = \limsup_{j \rightarrow \infty} \frac{1}{|T|} \int_{\mathcal{E}} d_T(\omega, x) dw_{n_j}(\omega, x) \text{ (by the definition of } w_{n_j}) \\ (10.6) & = \frac{1}{|T|} \int_{\mathcal{E}} d_T(\omega, x) d\mu(\omega, x) \text{ (as the sequence } \{w_{n_j} : j \in \mathbb{N}\} \text{ converges to } \mu). \end{aligned}$$

Now recall our assumption that $F_n \in \mathcal{T}_G, n \in \mathbb{N}$. By (10.6) we have

$$(10.7) \quad \limsup_{j \rightarrow \infty} \frac{1}{|F_{n_j}|} \int_{\mathcal{E}} d_{F_{n_j}}(\omega, x) d\nu_{n_j}(\omega, x) \leq \frac{1}{|F_n|} \int_{\mathcal{E}} d_{F_n}(\omega, x) d\mu(\omega, x)$$

for each $n \in \mathbb{N}$, from which (10.4) follows, once we take the infimum over all $n \in \mathbb{N}$. \square

11. ANOTHER VERSION OF THE LOCAL VARIATIONAL PRINCIPLE

Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a monotone sub-additive G -invariant family. When $G = \mathbb{Z}$ and $\mathbf{D} = \mathbf{D}^f$ for some $f \in \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$, Kifer [44] introduced the global fiber topological pressure using separated subsets with a positive constant ϵ and proved that the resulting pressure is the same if we use separated subsets with a positive random variable ϵ belonging to a natural class [44, Proposition 1.10]. Observe that each $(\Omega \times \mathcal{V})_{\mathcal{E}}$ with $\mathcal{V} \in \mathbf{C}_X^o$ is factor good, and thus it is easy to see that our definition recovers Kifer's definition of global pressure (using separated subsets with a positive constant ϵ). (The discussion is quite standard,

see for example [67, §7.2]). Hence, a natural question is whether, in analogy to [44, Proposition 1.10], is there a similar result for covers of not only a finite family but also a countable family in a natural class? This section is devoted to proving a result of this type.

Denote by $\mathfrak{C}_{\mathcal{E}}^o$ the set of all countable families $\mathcal{U} \subseteq (\mathcal{F} \times \mathcal{B}_X) \cap \mathcal{E}$ satisfying:

- (1) \mathcal{U} covers the whole space \mathcal{E} ,
- (2) $\mathcal{U}_{\omega} = \{U_{\omega} : U \in \mathcal{U}\} \in \mathfrak{C}_{\mathcal{E}_{\omega}}^o$ for \mathbb{P} -a.e. $\omega \in \Omega$ and
- (3) There exists an increasing sequence $\{\Omega_1 \subseteq \Omega_2 \subseteq \dots\} \subseteq \mathcal{F}$ such that $\lim_{n \rightarrow \infty} \mathbb{P}(\Omega_n) = 1$ and $\mathcal{U} \cap (\Omega_n \times X)$ is a finite family for each $n \in \mathbb{N}$.

Equation (3) may at first sight seem rather contrived. However, note that for a given a positive random variable ϵ , (3) is just the counterpart of the following basic fact:

$$\lim_{n \rightarrow \infty} \mathbb{P}(\{\omega \in \Omega : \epsilon(\omega) > \frac{1}{n}\}) = 1.$$

As we shall see, the class $\mathfrak{C}_{\mathcal{E}}^o$ plays a role in our setting analogous to that of the positive random variables in Kifer's setting.

Let $\mathcal{U} \in \mathfrak{C}_{\mathcal{E}}^o$. It is not hard to see that the function $N(\mathcal{U}, \omega)$ is measurable in $\omega \in \Omega$. Now let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ be a monotone sub-additive G -invariant family. The definitions and notation related to $\mathfrak{C}_{\mathcal{E}}^o$ can be extended to $\mathfrak{C}_{\mathcal{E}}^o$, including $P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F})$ for each $F \in \mathcal{F}_G$ and \mathbb{P} -a.e. $\omega \in \Omega$. In fact, let $F \in \mathcal{F}_G$, for \mathbb{P} -a.e. $\omega \in \Omega$ as in Proposition 5.3. Then we also have

$$(11.1) \quad P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}) = \min \left\{ \sum_{A(\omega) \in \alpha(\omega)} \sup_{x \in A(\omega)} e^{d_F(\omega, x)} : \alpha(\omega) \in \mathbf{P}((\mathcal{U}_F)_{\omega}) \right\}.$$

Moreover, for each $n \in \mathbb{N}$ set $\mathcal{U}_n = \mathcal{U} \cap (\Omega_n \times X) \cup \{(\Omega_n^c \times X)\} \cap \mathcal{E}$, then $\mathcal{U}_n \in \mathfrak{C}_{\mathcal{E}}^o$. It is now not hard to check that the sequence $\{P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}_n, \mathbf{F}) : n \in \mathbb{N}\}$ increases to $P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F})$ for \mathbb{P} -a.e. $\omega \in \Omega$. In particular, by Proposition 5.4 one has (observe that \mathbf{D} is monotone and hence non-negative):

- (1) for each $F \in \mathcal{F}_G$, the function $P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F})$ is measurable in $\omega \in \Omega$.

If, in addition, $\int_{\Omega} \log N(\mathcal{U}, \omega) d\mathbb{P}(\omega) < \infty$ then

- (2) $\{\log P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}) : F \in \mathcal{F}_G\}$ is a non-negative sub-additive G -invariant family in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and
- (3) for $p : \mathcal{F}_G \rightarrow \mathbb{R}, F \mapsto \int_{\Omega} \log P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}) d\mathbb{P}(\omega)$, p is a monotone non-negative G -invariant sub-additive function.

From this, we also introduce

$$P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) = \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\Omega} \log P_{\mathcal{E}}(\omega, \mathbf{D}, F_n, \mathcal{U}, \mathbf{F}) d\mathbb{P}(\omega).$$

With some standard arguments we can now introduce $h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U})$ for each $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$, and then similar to Proposition 5.6 it is easy to show

$$(11.2) \quad P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) \geq \sup_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D}).$$

All major results of the previous sections can now be carried out for the extended setting of this section. We single out only two of them as follows.

In the above notation, we have a local version of [44, Proposition 1.10].

Proposition 11.1. *Let $\mathcal{U} \in \mathfrak{C}_{\mathcal{E}}^{\circ}$ with the corresponding increasing sequence $\{\Omega_1 \subseteq \Omega_2 \subseteq \dots\} \subseteq \mathcal{F}$ satisfying $\lim_{n \rightarrow \infty} \mathbb{P}(\Omega_n) = 1$ and $\mathcal{U} \cap (\Omega_n \times X)$ is a finite family for each $n \in \mathbb{N}$. Define $\mathcal{U}_n, n \in \mathbb{N}$ as above. If $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ is a monotone sub-additive G -invariant family then*

$$(11.3) \quad \frac{P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F})}{P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}_n, \mathbf{F})} \leq \exp \sum_{g \in F} 1_{\Omega \setminus \Omega_n}(g\omega) \log N(\mathcal{U}, g\omega).$$

for each $F \in \mathcal{F}_G$, \mathbb{P} -a.e. $\omega \in \Omega$ and any $n \in \mathbb{N}$, and, additionally,

$$(11.4) \quad \lim_{n \rightarrow \infty} P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}_n, \mathbf{F}) = P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F})$$

when $\int_{\Omega} \log N(\mathcal{U}, \omega) d\mathbb{P}(\omega) < \infty$.

Before proving Proposition 11.1, we give the following remark.

Let (Z, s) be a metric space, For each $r > 0$ and any compact subset $Y \subseteq Z$, denote by $N_Y(r)$ the minimal number of closed balls of diameter r which cover Y .

Remark 11.2. *By the results from [44], given a continuous bundle RDS, $N_Y(r)$ is non-increasing and right continuous in $r > 0$ and is lower semi-continuous in Y on the space 2^X equipped with the Hausdorff topology. Further, for any positive random variable ϵ on $(\Omega, \mathcal{F}, \mathbb{P})$ the map $N_{\mathcal{E}_\omega}(\epsilon(\omega))$ is measurable in $\omega \in \Omega$. Based on this, Kifer defined the class \mathcal{N} by $\epsilon \in \mathcal{N}$ if and only if*

$$(11.5) \quad \int_{\Omega} \log N_{\mathcal{E}_\omega}(\epsilon(\omega)) d\mathbb{P}(\omega) < \infty.$$

He proved that the global pressure using separated subsets with a positive constant ϵ is the same if we used separated subsets with a positive random variable $\epsilon \in \mathcal{N}$ [44, Proposition 1.10] (by the compactness of the state space X obviously the positive constant must be contained in this class if it is viewed a constant function on $(\Omega, \mathcal{F}, \mathbb{P})$). Our assumption that $\int_{\Omega} \log N(\mathcal{U}, \omega) d\mathbb{P}(\omega) < \infty$ in Proposition 11.1 is just the analogue of (11.5) in our setting (and it is natural if we are to define $P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F})$ for $\mathcal{U} \in \mathfrak{C}_{\mathcal{E}}^{\circ}$). In fact, with the help of Proposition 11.1 (and its variation as in Remark 11.7) it is not hard to obtain [44, Proposition 1.10] using standard arguments as in [67, §7.2]. Here, we outline the basic ideas:

- (1) *If $\epsilon > 0$ is just a positive constant, and let $\mathcal{V}_1, \mathcal{V}_2 \in \mathbf{C}_X^{\circ}$ such that 2ϵ is a Lebesgue number of \mathcal{V}_1 and $\text{diam}(\mathcal{V}_2) < \epsilon$, where $\text{diam}(\mathcal{V}_2)$ denotes the maximal diameter of subsets $V_2 \in \mathcal{V}_2$, then it is straightforward to see:*

$$(11.6) \quad P_{\mathcal{E}}(\mathbf{D}, \mathcal{V}_1, \mathbf{F}) \leq P_{\mathcal{E}}(\mathbf{D}, \epsilon, \mathbf{F}) \leq P_{\mathcal{E}}(\mathbf{D}, \mathcal{V}_2, \mathbf{F}),$$

here $P_{\mathcal{E}}(\mathbf{D}, \epsilon, \mathbf{F})$ denotes the Kifer's pressure using separated subsets with the positive constant ϵ (for details see for example [44]). This implies that our definition recovers Kifer's definition of global pressure using separated subsets with a positive constant.

- (2) *Now if ϵ is a positive random variable satisfying (11.5), it is not hard to obtain another positive random variable $\epsilon_1 \leq \epsilon$ satisfying (11.5) such that ϵ_1 is the form of*

$$\epsilon_1 = \sum_{i \in I} a_i 1_{\Omega_i},$$

where I is a countable index, $a_i > 0$ for each $i \in I$ and $\{\Omega_i : i \in I\} \subseteq \mathcal{F}$ forms a countable partition of Ω (i.e. $\Omega_i \cap \Omega_j = \emptyset$ whenever $i \neq j$ for

$i, j \in I$ and $\bigcup_{i \in I} \Omega_i = \Omega$). From this it is easy to construct $\mathcal{V} \in \mathfrak{C}_{\mathcal{E}}^o$ such that

$$\mathcal{V} = \bigcup_{i \in I} \{\Omega_i \times \mathcal{V}_i\},$$

where $\mathcal{V}_i \in \mathbf{C}_X^o$ satisfies $\text{diam}(\mathcal{V}_i) < a_i$ for each $i \in I$. As in (11.6) one has

$$P_{\mathcal{E}}(\mathbf{D}, \epsilon, \mathbf{F}) \leq P_{\mathcal{E}}(\mathbf{D}, \mathcal{V}, \mathbf{F}),$$

and by Proposition 11.1 (and its variation as in Remark 11.7), we obtain [44, Proposition 1.10] in the setting of Kifer.

Now we prove Proposition 11.1.

Proof of Proposition 11.1. First we establish (11.3).

Let $F \in \mathcal{F}_G, \omega \in \Omega$ with $N(\mathcal{U}, g\omega)$ finite for each $g \in F$ and $n \in \mathbb{N}$ be fixed. Set

$$F^1 = \{g \in F : g\omega \in \Omega_n\} \text{ and } F^2 = \{g \in F : g\omega \in \Omega \setminus \Omega_n\} = F \setminus F^1.$$

By the construction of \mathcal{U}_n one has

$$\begin{aligned} & P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}_n, \mathbf{F}) \\ &= \inf \left\{ \sum_{A(\omega) \in \alpha(\omega)} \sup_{x \in A(\omega)} e^{d_F(\omega, x)} : \alpha(\omega) \in \mathbf{P}_{\mathcal{E}_\omega}, \alpha(\omega) \succeq ((\mathcal{U}_n)_F)_\omega \right\} \\ &= \inf \left\{ \sum_{A(\omega) \in \alpha(\omega)} \sup_{x \in A(\omega)} e^{d_F(\omega, x)} : \alpha(\omega) \in \mathbf{P}_{\mathcal{E}_\omega}, \right. \\ &\quad \left. \alpha(\omega) \succeq \bigvee_{g \in F} F_{g^{-1}, g\omega}(\mathcal{U}_n)_{g\omega} \right\} \text{ (using (4.6))} \\ &= \inf \left\{ \sum_{A(\omega) \in \alpha(\omega)} \sup_{x \in A(\omega)} e^{d_F(\omega, x)} : \alpha(\omega) \in \mathbf{P}_{\mathcal{E}_\omega}, \alpha(\omega) \succeq \bigvee_{g \in F^1} F_{g^{-1}, g\omega}(\mathcal{U}_n)_{g\omega} \right\} \\ (11.7) &= \inf \left\{ \sum_{A(\omega) \in \alpha(\omega)} \sup_{x \in A(\omega)} e^{d_F(\omega, x)} : \alpha(\omega) \in \mathbf{P}_{\mathcal{E}_\omega}, \alpha(\omega) \succeq \bigvee_{g \in F^1} F_{g^{-1}, g\omega} \mathcal{U}_{g\omega} \right\}. \end{aligned}$$

Moreover,

$$\begin{aligned}
& P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}, \mathbf{F}) \\
& \leq \inf \left\{ \sum_{A(\omega) \in \alpha(\omega), B(\omega) \in \beta(\omega)} \sup_{x \in A(\omega) \cap B(\omega)} e^{d_F(\omega, x)} : \right. \\
& \quad \left. \alpha(\omega) \in \mathbf{P}_{\mathcal{E}_\omega}, \alpha(\omega) \succeq \mathcal{U}_{F^1}, \beta(\omega) \in \mathbf{P}_{\mathcal{E}_\omega}, \beta(\omega) \succeq \mathcal{U}_{F^2} \right\} \\
& \leq \inf \left\{ \sum_{A(\omega) \in \alpha(\omega)} \sup_{x \in A(\omega)} e^{d_F(\omega, x)} : \alpha(\omega) \in \mathbf{P}_{\mathcal{E}_\omega}, \alpha(\omega) \succeq \mathcal{U}_{F^1} \right\} \\
& \quad \inf \left\{ \sum_{B(\omega) \in \beta(\omega)} 1 : \beta(\omega) \in \mathbf{P}_{\mathcal{E}_\omega}, \beta(\omega) \succeq \mathcal{U}_{F^2} \right\} \\
& \leq P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}_n, \mathbf{F}) \cdot N(\mathcal{U}_{F^2}, \omega) \text{ (using (4.6) and (11.7))} \\
& \leq P_{\mathcal{E}}(\omega, \mathbf{D}, F, \mathcal{U}_n, \mathbf{F}) \cdot \prod_{g \in F^2} N(\mathcal{U}, g\omega),
\end{aligned}$$

which implies the conclusion.

Next we prove (11.4). It is not hard to check that the sequence $\{P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}_n, \mathbf{F}) : n \in \mathbb{N}\}$ is increasing and each member is less than $P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F})$, that is,

$$(11.8) \quad P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) \geq \lim_{n \rightarrow \infty} P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}_n, \mathbf{F}).$$

For the other direction, by (11.3), for each $n \in \mathbb{N}$ we have

$$\begin{aligned}
& P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) \\
& \leq P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}_n, \mathbf{F}) + \limsup_{m \rightarrow \infty} \frac{1}{|F_m|} \int_{\Omega} \sum_{g \in F_m} 1_{\Omega \setminus \Omega_n}(g\omega) \log N(\mathcal{U}, g\omega) d\mathbb{P}(\omega) \\
(11.9) \quad & = P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}_n, \mathbf{F}) + \int_{\Omega} 1_{\Omega \setminus \Omega_n}(\omega) \log N(\mathcal{U}, \omega) d\mathbb{P}(\omega).
\end{aligned}$$

Now if $\int_{\Omega} \log N(\mathcal{U}, \omega) d\mathbb{P}(\omega) < \infty$, by the assumption that $\lim_{n \rightarrow \infty} \mathbb{P}(\Omega_n) = 1$ one has

$$\lim_{n \rightarrow \infty} \int_{\Omega} 1_{\Omega \setminus \Omega_n}(\omega) \log N(\mathcal{U}, \omega) d\mathbb{P}(\omega) = 0.$$

Hence, using (11.9),

$$P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) \leq \lim_{n \rightarrow \infty} P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}_n, \mathbf{F}).$$

Combined with (11.8), this proves the conclusion. \square

Thus we have the following general version of the local variational principle.

Theorem 11.3. *Let $\mathcal{U} \in \mathfrak{C}_{\mathcal{E}}^2$ with Ω_n and $\mathcal{U}_n, n \in \mathbb{N}$ as in Proposition 11.1. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space and each $\mathcal{U}_n, n \in \mathbb{N}$ is factor good. If $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ is a monotone sub-additive G -invariant family satisfying (\spadesuit) and $\int_{\Omega} \log N(\mathcal{U}, \omega) d\mathbb{P}(\omega) < \infty$ then*

$$P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) = \sup_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D})].$$

Proof. Obviously for each $n \in \mathbb{N}$ we have

$$\sup_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D})] \geq \sup_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}_n) + \mu(\mathbf{D})] = P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}_n, \mathbf{F}),$$

where the last identity follows from the assumptions and Theorem 7.1. Thus

$$\sup_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D})] \geq P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) \text{ (using Proposition 11.1).}$$

Combining with (11.2), we obtain the conclusion. \square

Question 11.4. Under the assumptions of Theorem 11.3, do we have

$$(11.10) \quad P_{\mathcal{E}}(\mathbf{D}, \mathcal{U}, \mathbf{F}) = \max_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}) + \mu(\mathbf{D})]?$$

Observe that in Theorem 7.1 (and its variation Theorem 10.2), the supremum can be realized as a maximum by direct construction.

Remark 11.5. By Theorem 6.9, one simple case when $\mathcal{U} \in \mathfrak{C}_{\mathcal{E}}^{\circ}$ satisfies the assumptions of Theorem 11.3 is: $\mathcal{U} \in \mathfrak{C}_{\mathcal{E}}^{\circ}$ has the form $\cup \{(A_i \times \mathcal{V}_i) \cap \mathcal{E} : i \in \mathbb{N}\}$ for $\{\mathcal{V}_i : i \in \mathbb{N}\} \subseteq \mathbf{C}_X^{\circ}$ and $\{A_i : i \in \mathbb{N}\} \subseteq \mathcal{F}$ with $A_i \cap A_j = \emptyset, i \neq j$ and $\bigcup_{i \in \mathbb{N}} A_i = \Omega$ satisfying $\sum_{i \in \mathbb{N}} \mathbb{P}(A_i) |\mathcal{V}_i| < \infty$.

Remark 11.6. We should remark that as in the discussions in §10, it is easy to see that if G admits a tiling Følner sequence then

- (1) Proposition 11.1 holds for a sub-additive G -invariant family $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ and
- (2) Theorem 11.3 holds for a sub-additive G -invariant family $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$ satisfying (\spadesuit) .

Remark 11.7. As commented in Remark 7.3, let $\mathcal{U} \in \mathfrak{C}_{\mathcal{E}}^{\circ}$ as in Proposition 11.1 and Theorem 11.3 with $\int_{\Omega} \log N(\mathcal{U}, \omega) d\mathbb{P}(\omega) < \infty$ and $f \in \mathbf{L}_{\mathcal{E}}^1(\Omega, C(X))$. Then

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\Omega} \log P_{\mathcal{E}}(\omega, \mathbf{D}^f, F_n, \mathcal{U}, \mathbf{F}) d\mathbb{P}(\omega) \\ &= \sup_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} [h_{\mu}^{(r)}(\mathbf{F}, \mathcal{U}) + \int_{\mathcal{E}} f(\omega, x) d\mu(\omega, x)]. \end{aligned}$$

Hence, in the case where G admits a tiling Følner sequence as in Remark 11.6, it equals the following limit (as in previous discussions, the limit must exist)

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{\Omega} \log P_{\mathcal{E}}(\omega, \mathbf{D}^f, F_n, \mathcal{U}, \mathbf{F}) d\mathbb{P}(\omega).$$

Part 3. Applications of the Local Variational Principle

In this part we give some interesting applications of the local variational principle established in previous sections. Namely, following the line of local entropy theory (see the recent survey [32] of Glasner and Ye and the references therein), we introduce and discuss both topological and measure-theoretical entropy tuples for a continuous bundle RDS. We then establish a variational relationship between them. We apply our results to obtain many known theorems and some new ones in local entropy theory.

12. ENTROPY TUPLES FOR A CONTINUOUS BUNDLE RDS

Following the line of local entropy theory (cf [32]), based on the local variational principle for fiber topological pressure established in previous sections, we introduce and discuss entropy tuples in both the topological and the measure-theoretical setting, for a given continuous bundle RDS, and establish a variational relation between them.

Let $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ and $(x_1, \dots, x_n) \in X^n \setminus \Delta_n(X)$, here $\Delta_n(X) = \{(x'_1, \dots, x'_n) : x'_1 = \dots = x'_n \in X\}, n \in \mathbb{N} \setminus \{1\}$. (x_1, \dots, x_n) is called a

- (1) *fiber topological entropy n -tuple of \mathbf{F}* if: For any $m \in \mathbb{N}$, there exists a closed neighborhood V_i of x_i of diameter at most $\frac{1}{m}$ for each $i = 1, \dots, n$ such that $\mathcal{V} \doteq \{V_1^c, \dots, V_n^c\} \in \mathbf{C}_X^o$ and $h_{\text{top}}^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V})_{\mathcal{E}}) > 0$. Equivalently, whenever V_i is a closed neighborhood of x_i for each $i = 1, \dots, n$ satisfying $\mathcal{V} \doteq \{V_1^c, \dots, V_n^c\} \in \mathbf{C}_X^o$ then $h_{\text{top}}^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V})_{\mathcal{E}}) > 0$.
- (2) *μ -fiber entropy n -tuple of \mathbf{F}* if: For any $m \in \mathbb{N}$, there exists a closed neighborhood V_i of x_i with diameter at most $\frac{1}{m}$ for each $i = 1, \dots, n$ such that $\mathcal{V} \doteq \{V_1^c, \dots, V_n^c\} \in \mathbf{C}_X^o$ and $h_{\mu}^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V})_{\mathcal{E}}) > 0$. Equivalently, whenever V_i is a closed neighborhood of x_i for each $i = 1, \dots, n$ satisfying $\mathcal{V} \doteq \{V_1^c, \dots, V_n^c\} \in \mathbf{C}_X^o$ then $h_{\mu}^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V})_{\mathcal{E}}) > 0$.

Denote by ${}_{\mathbb{P}}E_n^{(r)}(\mathcal{E}, G)$ (here we denote by \mathbb{P} the state system $(\Omega, \mathcal{F}, \mathbb{P}, G)$) and $E_{n,\mu}^{(r)}(\mathcal{E}, G)$ the set of all fiber topological entropy n -tuples of \mathbf{F} and μ -fiber entropy n -tuples of \mathbf{F} , respectively.

From the definitions, it is not hard to obtain:

Proposition 12.1. *Let $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ and $n \in \mathbb{N} \setminus \{1\}$. Then both ${}_{\mathbb{P}}E_n^{(r)}(\mathcal{E}, G) \cup \Delta_n(X)$ and $E_{n,\mu}^{(r)}(\mathcal{E}, G) \cup \Delta_n(X)$ are closed subsets of X^n .*

Before proceeding, we need:

Lemma 12.2. *Let $(Y, \mathcal{D}, \nu_n, G)$ be an MDS, $\mathcal{C} \subseteq \mathcal{D}$ a G -invariant sub- σ -algebra and $\alpha \in \mathbf{P}_Y$, where (Y, \mathcal{D}, ν_n) is a Lebesgue space, $n \in \mathbb{N}$. Assume that $0 \leq \lambda_n \leq 1, n \in \mathbb{N}$ satisfy $\sum_{n \in \mathbb{N}} \lambda_n = 1$. Then*

$$h_{\sum_{n \in \mathbb{N}} \lambda_n \nu_n}(G, \alpha|\mathcal{C}) = \sum_{n \in \mathbb{N}} \lambda_n h_{\nu_n}(G, \alpha|\mathcal{C}).$$

Proof. The case where there exist only finitely many $n \in \mathbb{N}$ with $\lambda_n > 0$ follows directly from Lemma 8.5. Now we consider the case where there exist infinitely many $n \in \mathbb{N}$ with $\lambda_n > 0$. Without loss of generality, we may assume $\lambda_n > 0$ for

each $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. Set $\lambda = \sum_{m=n+1}^{\infty} \lambda_m$ and $\nu = \sum_{m=n+1}^{\infty} \frac{\lambda_m}{\lambda} \nu_m$. Then (Y, \mathcal{D}, ν, G) is also an MDS, (Y, \mathcal{D}, ν) is also a Lebesgue space and $\sum_{n \in \mathbb{N}} \lambda_n \nu_n = \sum_{i=1}^n \lambda_i \nu_i + \lambda \nu$, and hence

$$\begin{aligned} \sum_{i=1}^n \lambda_i h_{\mu_i}(G, \alpha | \mathcal{C}) &\leq \sum_{i=1}^n \lambda_i h_{\mu_i}(G, \alpha | \mathcal{C}) + \lambda h_{\nu}(G, \alpha | \mathcal{C}) = h_{\sum_{n \in \mathbb{N}} \lambda_n \nu_n}(G, \alpha | \mathcal{C}) \\ &\leq \sum_{i=1}^n \lambda_i h_{\mu_i}(G, \alpha | \mathcal{C}) + |\alpha| \sum_{m=n+1}^{\infty} \lambda_m. \end{aligned}$$

Then the proof follows by letting $n \rightarrow \infty$ in the above inequalities. \square

Thus, we have the following variational relation between these two kinds of entropy tuples.

Theorem 12.3. *Let $n \in \mathbb{N} \setminus \{1\}$ and $0 < \lambda_1, \dots, \lambda_p < 1$ satisfy $\sum_{i=1}^p \lambda_i = 1, p \in \mathbb{N}$.*

- (1) *If $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ then $E_{n,\mu}^{(r)}(\mathcal{E}, G) \subseteq_{\mathbb{P}} E_n^{(r)}(\mathcal{E}, G)$.*
- (2) *Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space. Then*
 - (a) *if $\mu_1, \dots, \mu_p \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ then*

$$E_{n, \sum_{i=1}^p \lambda_i \mu_i}^{(r)}(\mathcal{E}, G) = \bigcup_{i=1}^p E_{n, \mu_i}^{(r)}(\mathcal{E}, G).$$

$$(b) \quad \mathbb{P} E_n^{(r)}(\mathcal{E}, G) = \bigcup_{\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} E_{n,\mu}^{(r)}(\mathcal{E}, G).$$

Proof. (1) is a direct corollary of Proposition 5.6. Now let us prove (2).

(2a) The containment \supseteq follows directly from Lemma 12.2. In fact, it is also easy to obtain the containment \subseteq from Lemma 12.2.

Set $\nu = \sum_{i=1}^p \lambda_i \mu_i$ and let $(x_1, \dots, x_n) \in E_{n,\nu}^{(r)}(\mathcal{E}, G)$. For any $m \in \mathbb{N}$ there exists a closed neighborhood V_i^m of x_i with diameter at most $\frac{1}{m}$ for each $i = 1, \dots, n$ such that $\mathcal{V}^m \doteq \{(V_1^m)^c, \dots, (V_n^m)^c\} \in \mathbf{C}_X^o$ and $h_{\nu}^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V}^m)_{\mathcal{E}}) > 0$, and so, by Lemma 12.2, $h_{\mu_j}^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V}^m)_{\mathcal{E}}) > 0$ for some $j \in \{1, \dots, p\}$. Clearly there exists $J \in \{1, \dots, p\}$ such that $h_{\mu_J}^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V}^m)_{\mathcal{E}}) > 0$ for infinitely many $m \in \mathbb{N}$, which implies $(x_1, \dots, x_n) \in E_{n,\mu_J}^{(r)}(\mathcal{E}, G)$.

(2b) Let $(x_1, \dots, x_n) \in \mathbb{P} E_n^{(r)}(\mathcal{E}, G)$. Then for any $m \in \mathbb{N}$ there exists a closed neighborhood V_i^m of x_i with diameter at most $\frac{1}{m}$ for each $i = 1, \dots, n$ such that $\mathcal{V}^m \doteq \{(V_1^m)^c, \dots, (V_n^m)^c\} \in \mathbf{C}_X^o$ and $h_{\text{top}}^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V}^m)_{\mathcal{E}}) > 0$. Observe that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space, using Proposition 6.9 one has that $(\Omega \times \mathcal{V}^m)_{\mathcal{E}} \in \mathbf{C}_{\mathcal{E}}^o$ is factor good and so by Theorem 7.1 there exists $\mu_m \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ such that $h_{\mu_m}^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V}^m)_{\mathcal{E}}) > 0$. Now set $\mu = \sum_{m \in \mathbb{N}} \frac{\mu_m}{2^m}$. Obviously, $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ and

$$h_{\mu}^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V}^m)_{\mathcal{E}}) \geq \frac{1}{2^m} h_{\mu_m}^{(r)}(\mathbf{F}, (\Omega \times \mathcal{V}^m)_{\mathcal{E}}) > 0$$

for each $m \in \mathbb{N}$ (using Lemma 12.2), which implies $(x_1, \dots, x_n) \in E_{n,\mu}^{(r)}(\mathcal{E}, G)$. Finally, combined with (1) we claim the conclusion. \square

In fact, we can prove:

Theorem 12.4. *There exists $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ with ${}_{\mathbb{P}}E_n^{(r)}(\mathcal{E}, G) = E_{n,\mu}^{(r)}(\mathcal{E}, G)$ for each $n \in \mathbb{N} \setminus \{1\}$.*

Proof. By Theorem 12.3, for each $n \in \mathbb{N} \setminus \{1\}$, there exists a dense sequence $\{(x_1^m, \dots, x_n^m) : m \in \mathbb{N}\} \subseteq {}_{\mathbb{P}}E_n^{(r)}(\mathcal{E}, G)$ with $(x_1^m, \dots, x_n^m) \in E_{n,\mu_n^m}^{(r)}(\mathcal{E}, G)$ for some $\mu_n^m \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$. Set

$$\mu = \sum_{n \in \mathbb{N} \setminus \{1\}} \frac{1}{2^{n-1}} \sum_{m \in \mathbb{N}} \frac{1}{2^m} \mu_n^m.$$

Obviously, $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$. By standard arguments used in the proof of Theorem 12.3, it is easy to check that $(x_1^m, \dots, x_n^m) \in E_{n,\mu}^{(r)}(\mathcal{E}, G)$ for each $n \in \mathbb{N} \setminus \{1\}$ and any $m \in \mathbb{N}$. Now using Proposition 12.1 by the selection of $(x_1^m, \dots, x_n^m), n \in \mathbb{N} \setminus \{1\}, m \in \mathbb{N}$ it is easy to claim that μ has the required property. \square

The following result tells us that both kinds of entropy tuples have the properties of lift and projection.

Proposition 12.5. *Let the family $\mathbf{F}_i = \{(F_i)_{g,\omega} : (\mathcal{E}_i)_{\omega} \rightarrow (\mathcal{E}_i)_{g\omega} | g \in G, \omega \in \Omega\}$ be a continuous bundle RDS over $(\Omega, \mathcal{F}, \mathbb{P}, G)$ with X_i the corresponding compact metric state space, $i = 1, 2$. Assume that $\pi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is a factor map from \mathbf{F}_1 to \mathbf{F}_2 and $n \in \mathbb{N} \setminus \{1\}, \mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}_1, G)$. If π is induced by a continuous surjection $\phi : X_1 \rightarrow X_2$ (i.e. $\pi : (\omega, x) \mapsto (\omega, \phi x)$), then*

- (1) $E_{n,\pi\mu}^{(r)}(\mathcal{E}_2, G) \subseteq (\phi \times \dots \times \phi)E_{n,\mu}^{(r)}(\mathcal{E}_1, G) \subseteq E_{n,\pi\mu}^{(r)}(\mathcal{E}_2, G) \cup \Delta_n(X_2)$.
- (2) ${}_{\mathbb{P}}E_n^{(r)}(\mathcal{E}_2, G) \subseteq (\phi \times \dots \times \phi){}_{\mathbb{P}}E_n^{(r)}(\mathcal{E}_1, G) \subseteq {}_{\mathbb{P}}E_n^{(r)}(\mathcal{E}_2, G) \cup \Delta_n(X_2)$.

Proof. As the proofs are similar, we shall only prove (1).

The proof follows the ideas of [4].

First, let $(x_1, \dots, x_n) \in E_{n,\mu}^{(r)}(\mathcal{E}_1, G)$ with $(\phi(x_1), \dots, \phi(x_n)) \in X_2^n \setminus \Delta_n(X_2)$. As $(x_1, \dots, x_n) \in E_{n,\mu}^{(r)}(\mathcal{E}_1, G)$, for any $M \in \mathbb{N}$ there exists a closed neighborhood V_i^M of x_i with diameter at most $\frac{1}{M}$ for each $i = 1, \dots, n$ such that $\mathcal{V}^M \doteq \{(V_1^M)^c, \dots, (V_n^M)^c\} \in \mathbf{C}_{X_1}^o$ and $h_{\mu}^{(r)}(\mathbf{F}_1, (\Omega \times \mathcal{V}^M)_{\mathcal{E}_1}) > 0$. Now let $m \in \mathbb{N}$ and say $V_i \subseteq X_2$ to be a closed neighborhood of $\phi(x_i)$ with diameter at most $\frac{1}{m}$ for each $i = 1, \dots, n$ such that $\mathcal{V} \doteq \{V_1^c, \dots, V_n^c\} \in \mathbf{C}_{X_2}^o$. By the continuity of ϕ , once M is sufficiently large, $\phi^{-1}V_i \supseteq V_i^M$ for each $i = 1, \dots, n$ and so

$$h_{\mu}^{(r)}(\mathbf{F}_1, \pi^{-1}(\Omega \times \mathcal{V})_{\mathcal{E}_2}) > 0$$

(observe that π is induced by ϕ and from the construction, one has $\pi^{-1}(\Omega \times \mathcal{V})_{\mathcal{E}_2} \supseteq (\Omega \times \mathcal{V}^M)_{\mathcal{E}_1}$), thus

$$h_{\pi\mu}^{(r)}(\mathbf{F}_2, (\Omega \times \mathcal{V})_{\mathcal{E}_2}) > 0$$

(using Lemma 6.8). This just means $(\phi x_1, \dots, \phi x_n) \in E_{n,\pi\mu}^{(r)}(\mathcal{E}_2, G)$.

Now let $(y_1, \dots, y_n) \in E_{n,\pi\mu}^{(r)}(\mathcal{E}_2, G)$. For any $m \in \mathbb{N}$ there exists a closed neighborhood V_i of y_i with diameter at most $\frac{1}{m}$ for each $i = 1, \dots, n$ such that $\mathcal{V} \doteq \{(V_1)^c, \dots, (V_n)^c\} \in \mathbf{C}_{X_2}^o$ and $h_{\pi\mu}^{(r)}(\mathbf{F}_2, (\Omega \times \mathcal{V})_{\mathcal{E}_2}) > 0$. For each $i = 1, \dots, n$, obviously we can cover $\phi^{-1}(V_i)$ by finite compact non-empty subsets $V_i^1, \dots, V_i^{k_i} \subseteq \phi^{-1}(V_i)$, $k_i \in \mathbb{N}$ with diameter at most $\frac{1}{m}$. For any $j_i = 1, \dots, k_i, i = 1, \dots, n$, set

$$\mathcal{W}_{j_1, \dots, j_n} = \{(\Omega \times V_i^{j_i})^c : i = 1, \dots, n\} \in \mathbf{C}_{\mathcal{E}_1}^o.$$

Observe that, for each $i = 1, \dots, n$,

$$(12.1) \quad (\Omega \times \phi^{-1}V_i)^c = \bigcap_{j=1}^{k_i} (\Omega \times V_i^j)^c.$$

First, by (12.1) one has

$$\pi^{-1}(\Omega \times \mathcal{V})_{\mathcal{E}_2} \preceq \bigvee_{j=1}^{k_1} \{(\Omega \times V_1^j)^c\} \cup \{(\Omega \times \phi^{-1}V_l)^c : l = 2, \dots, n\}$$

and so

$$\begin{aligned} 0 &< h_\mu^{(r)}(\mathbf{F}_1, \pi^{-1}(\Omega \times \mathcal{V})_{\mathcal{E}_2}) \quad (\text{using Lemma 6.8}) \\ &\leq h_\mu^{(r)}(\mathbf{F}_1, \bigvee_{j=1}^{k_1} \{(\Omega \times V_1^j)^c\} \cup \{(\Omega \times \phi^{-1}V_l)^c : l = 2, \dots, n\}) \\ &\leq \sum_{j=1}^{k_1} h_\mu^{(r)}(\mathbf{F}_1, \{(\Omega \times V_1^j)^c\} \cup \{(\Omega \times \phi^{-1}V_l)^c : l = 2, \dots, n\}), \end{aligned}$$

where, the last inequality uses Proposition 3.1, thus

$$h_\mu^{(r)}(\mathbf{F}_1, \{(\Omega \times V_1^{s_1})^c\} \cup \{(\Omega \times \phi^{-1}V_l)^c : l = 2, \dots, n\}) > 0$$

for some $s_1 \in \{1, \dots, k_1\}$. Now again by (12.1) one has

$$\{(\Omega \times V_1^{s_1})^c\} \cup \{(\Omega \times \phi^{-1}V_l)^c : l = 2, \dots, n\}$$

is coarser than

$$\bigvee_{j=1}^{k_2} \{(\Omega \times V_1^{s_1})^c\} \cup \{(\Omega \times V_2^j)^c\} \cup \{(\Omega \times \phi^{-1}V_l)^c : l = 3, \dots, n\},$$

similarly,

$$h_\mu^{(r)}(\mathbf{F}_1, \{(\Omega \times V_j^{s_j})^c : j = 1, 2\} \cup \{(\Omega \times \phi^{-1}V_l)^c : l = 3, \dots, n\}) > 0$$

for some $s_2 \in \{1, \dots, k_2\}$. After finitely many steps, one has

$$h_\mu^{(r)}(\mathbf{F}_1, \mathcal{W}_{s_1, \dots, s_n}) > 0$$

for some $s_j \in \{1, \dots, k_j\}, j = 1, \dots, n$. In conclusion, there exists $\{(W_i^m)^c : i = 1, \dots, n\} \in \mathbf{C}_{X_1}^o$ such that

- (a) $h_\mu^{(r)}(\mathbf{F}_1, \mathcal{U}^m) > 0$, where $\mathcal{U}^m = \{(\Omega \times W_i^m)^c : i = 1, \dots, n\}$ and
- (b) for each $i = 1, \dots, n$, both W_i^m and $\phi(W_i^m)$ have diameters at most $\frac{1}{m}$ and the distance between y_i and $\phi(W_i^m)$ is also at most $\frac{1}{m}$.

From (b), for each $i = 1, \dots, n$, $\{W_i^m : m \in \mathbb{N}\}$ converges to some point $x_i \in X_1$, moreover, it is obvious $\phi(x_i) = y_i$ (using (b) again, recall that $\phi : X_1 \rightarrow X_2$ is continuous). Our proof will be complete if we show that $(x_1, \dots, x_n) \in E_{n,\mu}^{(r)}(\mathcal{E}_1, G)$.

In fact, for any $p \in \mathbb{N}$ there exists a closed neighborhood W_i of x_i with diameter at most $\frac{1}{p}$ such that $\{W_1^c, \dots, W_n^c\} \in \mathbf{C}_{X_1}^o$. Obviously, once $m \in \mathbb{N}$ is sufficiently large, $W_i^m \subseteq W_i$ for each $i = 1, \dots, n$ and so $h_\mu^{(r)}(\mathbf{F}_1, \mathcal{W}) > 0$ where $\mathcal{W} = \{(\Omega \times W_i)^c : i = 1, \dots, n\}$ (using (a), observe $\mathcal{W} \succeq \mathcal{U}^m$). This implies $(x_1, \dots, x_n) \in E_{n,\mu}^{(r)}(\mathcal{E}_1, G)$, which completes the proof. \square

Moreover, we can show:

Proposition 12.6. *Let $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ and $n \in \mathbb{N} \setminus \{1\}$. Then*

- (1) $E_{n,\mu}^{(r)}(\mathcal{E}, G) \neq \emptyset$ if and only if $h_{\mu}^{(r)}(\mathbf{F}) > 0$.
- (2) ${}_{\mathbb{P}}E_n^{(r)}(\mathcal{E}, G) \neq \emptyset$ if and only if $h_{i_{\text{top}}}^{(r)}(\mathbf{F}) > 0$.

Proof. By similar arguments to that in the proof of Proposition 12.5, using Theorem 4.11 it is not hard to verify (1), and from the definitions it is not hard to obtain (2). (We omit the details). \square

By a *topological dynamical G -system (TDS)* (X, G) we mean that X is a compact metric space and G is a group of homeomorphisms of X with e_G acting as the identity.

As a direct corollary of Proposition 12.5, one has:

Proposition 12.7. *Let $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$. If \mathbf{F} is induced by a TDS (X, G) (i.e. $F_{g,\omega}$ is just the restriction of the action g over \mathcal{E}_{ω} for \mathbb{P} -a.e. $\omega \in \Omega$), then both $E_{n,\mu}^{(r)}(\mathcal{E}, G)$ and ${}_{\mathbb{P}}E_n^{(r)}(\mathcal{E}, G)$ are G -invariant subsets of X^n .*

Let $(x_1, \dots, x_n) \in X^n \setminus \Delta_n(X)$, $n \in \mathbb{N} \setminus \{1\}$. (x_1, \dots, x_n) is called a *fiber n -tuple* of \mathbf{F} if for any $m \in \mathbb{N}$ there exist $\Omega^* \in \mathcal{F}$ and a closed neighborhood V_i of x_i with diameter at most $\frac{1}{m}$ for each $i = 1, \dots, n$ such that $\mathcal{V} = \{V_1^c, \dots, V_n^c\} \in \mathbf{C}_X^o$, $\mathbb{P}(\Omega^*) > 0$ and $\prod_{i=1}^n \{\omega\} \times V_i \cap \mathcal{E}^n \neq \emptyset$ for each $\omega \in \Omega^*$. Denote by ${}_{\mathbb{P}}E_n^{(r)}(\mathcal{E})$ the set of all fiber n -tuples of \mathbf{F} . It may happen ${}_{\mathbb{P}}E_n^{(r)}(\mathcal{E}) = \emptyset$: for example, \mathcal{E}_{ω} is just a singleton for \mathbb{P} -a.e. $\omega \in \Omega$.

As in Proposition 12.1, we have (combining with our definition):

Proposition 12.8. *Let $n \in \mathbb{N} \setminus \{1\}$. Then ${}_{\mathbb{P}}E_n^{(r)}(\mathcal{E}) \cup \Delta_n(X) \subseteq \overline{\bigcup_{\omega \in \Omega} \mathcal{E}_{\omega}^n} \cup \Delta_n(X)$ is a closed subset. Moreover, if \mathbf{F} is induced by a TDS (X, G) then the subset ${}_{\mathbb{P}}E_n^{(r)}(\mathcal{E})$ is G -invariant.*

As in the proof of Proposition 12.5, we obtain:

Proposition 12.9. *Let the family $\mathbf{F}_i = \{(F_i)_{g,\omega} : (\mathcal{E}_i)_{\omega} \rightarrow (\mathcal{E}_i)_{g\omega} | g \in G, \omega \in \Omega\}$ be a continuous bundle RDS over $(\Omega, \mathcal{F}, \mathbb{P}, G)$ with X_i the corresponding compact metric state space, $i = 1, 2$. Assume that $\pi : \mathcal{E}_1 \rightarrow \mathcal{E}_2$ is a factor map from \mathbf{F}_1 to \mathbf{F}_2 and $n \in \mathbb{N} \setminus \{1\}$. If π is induced by a continuous surjection $\phi : X_1 \rightarrow X_2$, then*

$${}_{\mathbb{P}}E_n^{(r)}(\mathcal{E}_2) \subseteq (\phi \times \dots \times \phi)({}_{\mathbb{P}}E_n^{(r)}(\mathcal{E}_1)) \subseteq {}_{\mathbb{P}}E_n^{(r)}(\mathcal{E}_2) \cup \Delta_n(X_2).$$

Before proceeding, we observe:

Lemma 12.10. *Let $V_1, \dots, V_n \in \mathcal{B}_X$, $n \in \mathbb{N} \setminus \{1\}$. Then*

$$\Omega(V_1, \dots, V_n) \doteq \{\omega \in \Omega : \prod_{i=1}^n \{\omega\} \times V_i \cap \mathcal{E}^n = \emptyset\} \in \mathcal{F}.$$

Proof. Assume that $\pi : \Omega \times X \rightarrow \Omega$ is the natural projection. Using Lemma 4.2, we have

$$\Omega_0 \doteq \{\omega \in \Omega : \prod_{i=1}^n \{\omega\} \times V_i \cap \mathcal{E}^n \neq \emptyset\} = \bigcap_{i=1}^n \pi(\Omega \times V_i \cap \mathcal{E}) \in \mathcal{F}.$$

Observe $\Omega_0 = \Omega \setminus \Omega(V_1, \dots, V_n)$, one has $\Omega(V_1, \dots, V_n) \in \mathcal{F}$. \square

Lemma 12.11. *Let $\Omega^* \in \mathcal{F}$ and $\mathcal{V} = \{V_1^c, \dots, V_n^c\} \in \mathbf{C}_X, n \in \mathbb{N} \setminus \{1\}$. Set $\mathcal{U} = \{(\Omega^* \times V_i)^c : i = 1, \dots, n\}$ and $\mathcal{U}' = \{(\Omega' \times V_i)^c : i = 1, \dots, n\}$, where $\Omega' = \Omega^* \setminus \Omega(V_1, \dots, V_n)$. Then*

- (1) $\mathcal{U} \succeq \mathcal{U}'$ and $\mathcal{U}_\omega \supseteq \mathcal{U}'_\omega$, (and hence $\mathcal{U}'_\omega \succeq \mathcal{U}_\omega$) for each $\omega \in \Omega$.
- (2) $h_{top}^{(r)}(\mathbf{F}, \mathcal{U}) = h_{top}^{(r)}(\mathbf{F}, \mathcal{U}')$. In particular, if $h_{top}^{(r)}(\mathbf{F}, \mathcal{U}) > 0$ then $\mathbb{P}(\Omega') > 0$.
- (3) if $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ then $h_\mu^{(r)}(\mathbf{F}, \mathcal{U}) = h_\mu^{(r)}(\mathbf{F}, \mathcal{U}'), h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}) = h_{\mu,+}^{(r)}(\mathbf{F}, \mathcal{U}')$.

Proof. (1) We only need check that $\mathcal{E}_\omega \in \mathcal{U}_\omega$ for each $\omega \in \Omega(V_1, \dots, V_n)$ as, from the construction of \mathcal{U}' , it is clear that $\{\mathcal{E}_\omega\} = \mathcal{U}'_\omega$ for each $\omega \in \Omega(V_1, \dots, V_n)$. In fact, if $\omega \in \Omega(V_1, \dots, V_n)$ then $\{\omega\} \times V_i \cap \mathcal{E} = \emptyset$ for some $i \in \{1, \dots, n\}$, which implies $\mathcal{E}_\omega \subseteq V_i^c$, particularly, $\mathcal{E}_\omega \in \mathcal{U}_\omega$.

Combining Proposition 3.1, Lemma 4.8, Proposition 5.8 and the definitions, both (2) and (3) follow directly from (1). \square

Thus, we have:

Proposition 12.12. *Let $(x_1, \dots, x_n) \in X^n \setminus \Delta_n(X), n \in \mathbb{N} \setminus \{1\}$. Then*

- (1) $(x_1, \dots, x_n) \in \mathbb{P} E_n^{(r)}(\mathcal{E})$ if and only if whenever V_i is a closed neighborhood of x_i for each $i = 1, \dots, n$ such that $\{V_1^c, \dots, V_n^c\} \in \mathbf{C}_X^o$ then $\mathbb{P}(\Omega(V_1, \dots, V_n)) < 1$.
- (2) $\mathbb{P} E_n^{(r)}(\mathcal{E}, G) \subseteq \mathbb{P} E_n^{(r)}(\mathcal{E})$.
- (3) Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space. Then $(x_1, \dots, x_n) \in E_{n,\mu}^{(r)}(\mathcal{E}, G)$ if and only if whenever V_i is a closed neighborhood of x_i for each $i = 1, \dots, n$ such that $\{V_1^c, \dots, V_n^c\} \in \mathbf{C}_X^o$ then there exists $\Omega^* \in \mathcal{F}$ such that $h_\mu^{(r)}(\mathbf{F}, \alpha) > 0$ for each $\alpha \in \mathbf{P}_{\mathcal{E}}$ satisfying $\alpha \succeq \mathcal{U}$, where $\mathcal{U} = \{(\Omega^* \times V_i)^c : i = 1, \dots, n\}$.

Proof. (1), (2) and (3) follow from Lemma 12.10, Lemma 12.11 and Theorem 3.13, respectively. \square

In the remainder of this section, we equip with $(\Omega, \mathcal{F}, \mathbb{P})$ the structure of a topological space. Before proceeding, we need some preparations.

Let Y be a topological space and ν a probability measure over (Y, \mathcal{B}_Y) . Denote by $\text{supp}(\nu)$ the set of all points $y \in Y$ such that $\nu(V) > 0$ whenever V is an open neighborhood of y . Thus, $\text{supp}(\nu) \subseteq Y$ is a closed subset.

Observe that if Ω is a topological space with $\mathcal{F} = \mathcal{B}_\Omega$, then each $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ may be viewed as a Borel probability measure over the topological space $\Omega \times X$. From the definition, it is easy to check:

Lemma 12.13. *Let $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ and $n \in \mathbb{N} \setminus \{1\}$. Assume that Ω is a topological space with $\mathcal{F} = \mathcal{B}_\Omega$. Then $\text{supp}(\lambda_n^{\mathcal{F}\mathcal{E}}(\mu)) \subseteq \text{supp}(\mu)^n \subseteq (\text{supp}(\mathbb{P}) \times X)^n$.*

We also have:

Lemma 12.14. *Let $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ and $((\omega_1, x_1), \dots, (\omega_n, x_n)) \in \text{supp}(\lambda_n^{\mathcal{F}\mathcal{E}}(\mu)), n \in \mathbb{N} \setminus \{1\}$. Assume that Ω is a Hausdorff space with $\mathcal{F} = \mathcal{B}_\Omega$. Then $\omega_1 = \dots = \omega_n$.*

Proof. From the definitions, it is easy to see that, whenever $A_i \in (\mathcal{F} \times \mathcal{B}_X) \cap \mathcal{E}$ satisfies $A_i \subseteq \Omega_i \times X$ for some $\Omega_i \in \mathcal{F}$ (for each $i = 1, \dots, n$), observe that

$$(\Omega_i \times X) \cap \mathcal{E} \in \mathcal{F}_{\mathcal{E}} \subseteq \mathcal{P}^{\mathcal{F}\mathcal{E}}(\mathcal{E}, (\mathcal{F} \times \mathcal{B}_X) \cap \mathcal{E}, \mu, G)$$

for each $i = 1, \dots, n$, and so

$$\begin{aligned} \lambda_n^{\mathcal{F}\mathcal{E}}(\mu)\left(\prod_{i=1}^n A_i\right) &= \int_{\mathcal{E}} \prod_{i=1}^n \mu(A_i | \mathcal{P}^{\mathcal{F}\mathcal{E}}(\mathcal{E}, (\mathcal{F} \times \mathcal{B}_X) \cap \mathcal{E}, \mu, G)) d\mu \\ &\leq \int_{\mathcal{E}} \prod_{i=1}^n \mu((\Omega_i \times X) \cap \mathcal{E} | \mathcal{P}^{\mathcal{F}\mathcal{E}}(\mathcal{E}, (\mathcal{F} \times \mathcal{B}_X) \cap \mathcal{E}, \mu, G)) d\mu \\ &= \int_{\mathcal{E}} \prod_{i=1}^n 1_{(\Omega_i \times X) \cap \mathcal{E}} d\mu = \mu\left(\left(\bigcap_{i=1}^n \Omega_i \times X\right) \cap \mathcal{E}\right) = \mathbb{P}\left(\bigcap_{i=1}^n \Omega_i\right), \end{aligned}$$

hence if $\mathbb{P}(\bigcap_{i=1}^n \Omega_i) = 0$ then $\lambda_n^{\mathcal{F}\mathcal{E}}(\mu)(\prod_{i=1}^n A_i) = 0$.

Now, for $((\omega_1, x_1), \dots, (\omega_n, x_n)) \in \mathcal{E}^n$, if $\omega_i \neq \omega_j$ for some $1 \leq i < j \leq n$ then obviously there exist open neighborhoods Ω_i (Ω_j , respectively) of x_i (x_j , respectively) such that $\Omega_i \cap \Omega_j = \emptyset$. Thus

$$\lambda_n^{\mathcal{F}\mathcal{E}}(\mu)\left(\prod_{k \in \{1, \dots, n\} \setminus \{i, j\}} (\Omega \times X) \cap \mathcal{E} \times \prod_{p=i, j} (\Omega_p \times X) \cap \mathcal{E}\right) = 0,$$

which implies $((\omega_1, x_1), \dots, (\omega_n, x_n)) \notin \text{supp}(\lambda_n^{\mathcal{F}\mathcal{E}}(\mu))$. This finishes our proof. \square

Hence one has:

Theorem 12.15. *Let $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ and $(x_1, \dots, x_n) \in X^n \setminus \Delta_n(X)$, $n \in \mathbb{N} \setminus \{1\}$. Then*

- (1) *Both (a) and (c) imply (b).*
- (2) *If Ω is a Polish space with $\mathcal{F} = \mathcal{B}_{\Omega}$ then (a) \iff (b).*
- (3) *If Ω is a compact metric space with $\mathcal{F} = \mathcal{B}_{\Omega}$ then (a) \iff (b) \iff (c).*

Where

- (a) $(x_1, \dots, x_n) \in E_{n, \mu}^{(r)}(\mathcal{E}, G)$.
- (b) *If V_i is a Borel neighborhood of x_i for each $i = 1, \dots, n$ then*

$$\lambda_n^{\mathcal{F}\mathcal{E}}(\mu)\left(\prod_{i=1}^n \Omega \times V_i \cap \mathcal{E}^n\right) > 0.$$

- (c) *There exists $\omega \in \Omega$ such that $((\omega, x_1), \dots, (\omega, x_n)) \in \text{supp}(\lambda_n^{\mathcal{F}\mathcal{E}}(\mu))$.*

Proof. (1) (a) \implies (b) follows from Lemma 3.11, and (c) obviously implies (b).

(2) If Ω is a Polish space with $\mathcal{F} = \mathcal{B}_{\Omega}$ then $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space, and so (b) \implies (a) follows from the definitions and Theorem 3.13. Hence, combining with (1), one has (a) \iff (b).

(3) Now assume that Ω is a compact metric space with $\mathcal{F} = \mathcal{B}_{\Omega}$. By (1) and (2), it remains to show (b) \implies (c).

For each $\omega \in \Omega$ and $r > 0$ denote by $B(\omega, r)$ the open ball of Ω with center ω and radius r . For any $m \in \mathbb{N}$, let V_i^m be a Borel neighborhood of x_i with diameter at most $\frac{1}{m}$ for each $i = 1, \dots, n$. By the assumption that

$$\lambda_n^{\mathcal{F}\mathcal{E}}(\mu)\left(\prod_{i=1}^n \Omega \times V_i^m \cap \mathcal{E}^n\right) > 0.$$

Observe that Ω is a compact metric space. Obviously $\Omega_m \neq \emptyset$, where

$$\Omega_m = \{(\omega_1, \dots, \omega_n) \in \Omega^n : \lambda_n^{\mathcal{F}^\varepsilon}(\mu) \left(\prod_{i=1}^n B(\omega_i, \frac{1}{m}) \times V_i^m \cap \mathcal{E}^n \right) > 0\}.$$

Set $\Omega^* = \bigcap_{m \in \mathbb{N}} \overline{\Omega_m}$. Then $\Omega^* \subseteq \Omega^n$ is a non-empty subset, as Ω^n is a compact metric space and for each $m_1 \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that if $m \geq M$ then $\Omega_m \subseteq \Omega_{m_1}$. Moreover, whenever $(\omega_1, \dots, \omega_n) \in \Omega^*$ then $((\omega_1, x_1), \dots, (\omega_n, x_n)) \in \text{supp}(\lambda_n^{\mathcal{F}^\varepsilon}(\mu))$ (and hence, using Lemma 12.14, $\omega_1 = \dots = \omega_n$): in fact, for any $(\omega_1, \dots, \omega_n) \in \Omega^*$ let V be a Borel neighborhood of $((\omega_1, x_1), \dots, (\omega_n, x_n))$. It is clear that once $m \in \mathbb{N}$ is large enough, there exists $(\omega_1^m, \dots, \omega_n^m) \in \Omega_m$ such that, if V_i is the closed ball in X with center x_i and radius $\frac{1}{m}$ for each $i = 1, \dots, n$ then $\prod_{i=1}^n B(\omega_i, \frac{1}{m}) \times V_i \subseteq V$, and hence $\lambda_n^{\mathcal{F}^\varepsilon}(\mu)(V) > 0$. This finishes the proof. \square

As a direct corollary of Theorem 12.3 and Theorem 12.15, one has:

Theorem 12.16. *Let $\mu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)$ and $n \in \mathbb{N} \setminus \{1\}$ with $\pi_n : (\Omega \times X)^n \rightarrow X^n$ the natural projection. Assume that Ω is a compact metric space with $\mathcal{F} = \mathcal{B}_\Omega$. Then*

$$E_{n,\mu}^{(r)}(\mathcal{E}, G) = \pi_n(\text{supp}(\lambda_n^{\mathcal{F}^\varepsilon}(\mu))) \setminus \Delta_n(X),$$

$$\mathbb{P}E_n^{(r)}(\mathcal{E}, G) = \pi_n \left(\bigcup_{\nu \in \mathcal{P}_{\mathbb{P}}(\mathcal{E}, G)} \text{supp}(\lambda_n^{\mathcal{F}^\varepsilon}(\nu)) \right) \setminus \Delta_n(X).$$

13. APPLICATIONS TO A GENERAL TOPOLOGICAL DYNAMICAL SYSTEM

In this section, we apply results obtained in the previous sections to the case of a Topological Dynamical System (TDS). We recover many recent results in the local entropy theory of \mathbb{Z} -actions (cf [4, 6, 32, 34, 36, 37]) and of infinite countable discrete amenable group actions (cf [37]). We also prove new results, some of which are novel even in the case of infinite countable discrete amenable groups, for example Theorem 13.1, Theorem 13.4 etc.

Let (Y, G) be a TDS. Denote by $\mathcal{P}(Y, G)$ the set of all G -invariant elements from $\mathcal{P}(Y)$ which we suppose equipped with the weak star topology. Then $\mathcal{P}(Y, G)$ is a non-empty compact metric space and, for each $\nu \in \mathcal{P}(Y)$, $(Y, \mathcal{B}_Y^\nu, \nu)$ (also denoted by (Y, \mathcal{B}_Y, ν) if there is no any ambiguity) is a Lebesgue space, where \mathcal{B}_Y^ν is the ν -completion of \mathcal{B}_Y .

Recall that $\pi : (Y_1, G) \rightarrow (Y_2, G)$ is a *factor map from TDS (Y_1, G) to TDS (Y_2, G)* if $\pi : Y_1 \rightarrow Y_2$ is a continuous surjection compatible with the actions of G (i.e. $\pi \circ g(y_1) = g \circ \pi(y_1)$ for each $g \in G$ and any $y_1 \in Y_1$).

Let $\pi : (Y_1, G) \rightarrow (Y_2, G)$ be a factor map between TDSs and $\mathcal{W} \in \mathbf{C}_{Y_1}, \nu_1 \in \mathcal{P}(Y_1, G)$. Observe that the sub- σ -algebra $\pi^{-1}\mathcal{B}_{Y_2} \subseteq \mathcal{B}_{Y_1}$ is G -invariant, so we may introduce the *measure-theoretic ν_1 -entropy of \mathcal{W} relative to π* by

$$h_{\nu_1}(G, \mathcal{W}|\pi) = h_{\nu_1}(G, \mathcal{W}|\pi^{-1}\mathcal{B}_{Y_2}) = h_{\nu_1,+}(G, \mathcal{W}|\pi^{-1}\mathcal{B}_{Y_2}),$$

where the second equality follows from Theorem 3.3, since $(Y_1, \mathcal{B}_{Y_1}, \nu_1)$ is always a Lebesgue space. Finally, the *measure-theoretic ν_1 -entropy of (Y_1, G) relative to π* may be defined as

$$h_{\nu_1}(G, Y_1|\pi) = h_{\nu_1}(G, Y_1|\pi^{-1}\mathcal{B}_{Y_2}).$$

Now assume that $\mathcal{W} \in \mathbf{C}_{Y_1}^o$. For each $y_2 \in Y_2$ let $N(\mathcal{W}, \pi^{-1}y_2)$ be the minimal cardinality of a sub-family of \mathcal{W} covering $\pi^{-1}(y_2)$ and put

$$N(\mathcal{W}|\pi) = \sup_{y_2 \in Y_2} N(\mathcal{W}, \pi^{-1}y_2).$$

It is easy to see that

$$\log N(\mathcal{W}_\bullet|\pi) : \mathcal{F}_G \rightarrow \mathbb{R}, F \mapsto \log N(\mathcal{W}_F|\pi)$$

is a monotone non-negative G -invariant sub-additive function, and so by Proposition 2.2 we may define the *topological entropy of \mathcal{W} relative to π* by

$$h_{\text{top}}(G, \mathcal{W}|\pi) = \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \log N(\mathcal{W}_{F_n}|\pi).$$

Last, the *topological entropy of (Y_1, G) relative to π* may be introduced as

$$h_{\text{top}}(G, Y_1|\pi) = \sup_{\mathcal{W} \in \mathbf{C}_{Y_1}^o} h_{\text{top}}(G, \mathcal{W}|\pi).$$

In fact, more generally, given a monotone sub-additive G -invariant family $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq C(Y_1)$, where $C(Y_1)$ denotes the space of all real-valued continuous functions on Y_1 , and the concepts of monotonicity, sub-additivity and G -invariance for functions are introduced similarly, we can introduce

$$P_\pi(y_2, \mathbf{D}, F, \mathcal{W}) = \inf \left\{ \sum_{A \in \alpha} \sup_{x \in A \cap \pi^{-1}(y_2)} e^{d_F(x)} : \alpha \in \mathbf{P}_{Y_1}, \alpha \succeq \mathcal{W}_F \right\}$$

for any $y_2 \in Y_2$ and each $F \in \mathcal{F}_G$ and

$$P_\pi(\mathbf{D}, \mathcal{W}) = \lim_{n \rightarrow \infty} \frac{1}{F_n} \sup_{y_2 \in Y_2} \log P_\pi(y_2, \mathbf{D}, F, \mathcal{W}).$$

It is not hard to check that these concepts are well-defined. We may further define

$$P_\pi(\mathbf{D}) = \sup_{\mathcal{U} \in \mathbf{C}_{Y_1}^o} P_\pi(\mathbf{D}, \mathcal{U}).$$

Let $\pi : (Y_1, G) \rightarrow (Y_2, G)$ be a factor map between TDSs, $\nu_1 \in \mathcal{P}(Y_1, G)$ and $(x_1, \dots, x_n) \in Y_1^n \setminus \Delta_n(Y_1)$, $n \in \mathbb{N} \setminus \{1\}$. (x_1, \dots, x_n) is called a:

- (1) *relative topological entropy n -tuple relevant to π* if for any $m \in \mathbb{N}$ there exists a closed neighborhood V_i of x_i with diameter at most $\frac{1}{m}$ for each $i = 1, \dots, n$ such that $\mathcal{V} \doteq \{V_1^c, \dots, V_n^c\} \in \mathbf{C}_{Y_1}^o$ and $h_{\text{top}}(G, \mathcal{V}|\pi) > 0$.
- (2) *relative measure-theoretical ν_1 -entropy n -tuple relevant to π* if for any $m \in \mathbb{N}$ there exists a closed neighborhood V_i of x_i with diameter at most $\frac{1}{m}$ for each $i = 1, \dots, n$ such that $\mathcal{V} \doteq \{V_1^c, \dots, V_n^c\} \in \mathbf{C}_{Y_1}^o$ and $h_{\nu_1}(G, \mathcal{V}|\pi) > 0$.

Denote by $E_n(Y_1, G|\pi)$ and $E_n^{\nu_1}(Y_1, G|\pi)$ the set of all relative topological entropy n -tuples relevant to π and relative measure-theoretical ν_1 -entropy n -tuples relevant to π , respectively. Remark that these definitions recover the definitions of these terms introduced in [4, 6, 34, 36, 37].

Now let $\pi : (Y_1, G) \rightarrow (Y_2, G)$ be a factor map between TDSs, $\nu_2 \in \mathcal{P}(Y_2, G)$, $\mathcal{V} \in \mathbf{C}_{Y_1}$ and $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq C(Y_1)$ a monotone sub-additive G -invariant family. For each $g \in G$ and any $y_2 \in Y_2$, set

$$F_{g, y_2}^\pi : \{y_2\} \times \pi^{-1}(y_2) \rightarrow \{gy_2\} \times \pi^{-1}(gy_2), (y_2, y_1) \mapsto (gy_2, gy_1)$$

and

$$\mathcal{E}_\pi = \{(y_2, y_1) \in Y_2 \times Y_1 : \pi(y_1) = y_2\}.$$

It is easy to see that \mathcal{E}_π is a non-empty compact subset of $Y_2 \times Y_1$, and G acts naturally on \mathcal{E}_π . One checks that the family

$$\mathbf{F}^\pi \doteq \{F_{g,y_2}^\pi : \{y_2\} \times \pi^{-1}(y_2) \rightarrow \{gy_2\} \times \pi^{-1}(gy_2) | g \in G, y_2 \in Y_2\}$$

forms a continuous bundle RDS over MDS $(Y_2, \mathcal{B}_{Y_2}, \nu_2, G)$ with $(Y_2, \mathcal{B}_{Y_2}, \nu_2)$ a Lebesgue space, and the family \mathbf{D} may be viewed as a monotone sub-additive G -invariant family $\mathbf{D}^\pi = \{d_F^\pi : F \in \mathcal{F}_G\} \subseteq \mathbf{L}_{\mathcal{E}_\pi}^1(Y_2, C(Y_1))$ by a natural map

$$d_F^\pi(y_2, y_1) = d_F(y_1) \text{ for any } (y_2, y_1) \in \mathcal{E}_\pi.$$

For each $V \in \mathcal{V}$, we can introduce

$$(13.1) \quad V^\pi = \{(\pi y_1, y_1) : y_1 \in V\} = Y_2 \times V \cap \mathcal{E}_\pi,$$

and so

$$(13.2) \quad \mathcal{V}^\pi \doteq \{V^\pi : V \in \mathcal{V}\} \in \mathbf{C}_{\mathcal{E}_\pi}.$$

In fact, if $\mathcal{V} \in \mathbf{C}_{Y_1}^o$ then it is simple to see that $\mathcal{V}^\pi \in \mathbf{C}_{\mathcal{E}_\pi}^o$. From now on, for the state space $(Y_2, \mathcal{B}_{Y_2}, \nu_2, G)$ we shall denote

$$\nu_2 h_{\text{top}}^{(r)}(\mathbf{F}^\pi, \mathcal{V}^\pi), \nu_2 h_{\text{top}}^{(r)}(\mathbf{F}^\pi), \nu_2 P_{\mathcal{E}_\pi}(\mathbf{D}^\pi, \mathcal{V}^\pi, \mathbf{F}^\pi), \nu_2 P_{\mathcal{E}_\pi}(\mathbf{D}^\pi, \mathbf{F}^\pi)$$

as the fiber topological entropy of \mathbf{F}^π (with respect to \mathcal{V}^π) and the fiber topological \mathbf{D}^π -pressure of \mathbf{F}^π (with respect to \mathcal{V}^π), respectively.

Moreover, there is a natural one-to-one map between $\mathcal{P}_{\nu_2}(\mathcal{E}_\pi, G)$ and

$$\{\nu_1 \in \mathcal{P}(Y_1, G) : \pi\nu_1 = \nu_2\} \text{ (denoted by } \mathcal{P}_{\nu_2}(Y_1, G)),$$

a non-empty compact subset of $\mathcal{P}(Y_1, G)$, as \mathcal{E}_π is identical to Y_1 by the natural homeomorphism $(y_2, y_1) \mapsto y_1$; similarly, there is a natural one-to-one map between $\mathcal{P}_{\nu_2}(\mathcal{E}_\pi)$ and

$$\{\nu_1 \in \mathcal{P}(Y_1) : \pi\nu_1 = \nu_2\} \text{ (denoted by } \mathcal{P}_{\nu_2}(Y_1)),$$

which extends the one-to-one map between $\mathcal{P}_{\nu_2}(\mathcal{E}_\pi, G)$ and $\mathcal{P}_{\nu_2}(Y_1, G)$. In fact, given a sequence $\{\nu_1^n : n \in \mathbb{N}\} \subseteq \mathcal{P}_{\nu_2}(\mathcal{E}_\pi)$ and $\nu_1 \in \mathcal{P}_{\nu_2}(\mathcal{E}_\pi)$, if $\mu_1^n, n \in \mathbb{N}, \mu_1$ is the natural correspondence of $\nu_1^n, n \in \mathbb{N}, \nu_1$ in $\mathcal{P}_{\nu_2}(Y_1)$, respectively, then it is not hard to check that the following statements are equivalent:

- (1) the sequence $\{\nu_1^n : n \in \mathbb{N}\}$ converges to ν_1 ;
- (2) the sequence $\{\int_{Y_2 \times Y_1} f d\nu_1^n : n \in \mathbb{N}\}$ converges to $\int_{Y_2 \times Y_1} f d\nu_1$ for any $f \in C(Y_2 \times Y_1)$;
- (3) the sequence $\{\int_{\mathcal{E}_\pi} f d\nu_1^n : n \in \mathbb{N}\}$ converges to $\int_{\mathcal{E}_\pi} f d\nu_1$ for any $f \in C(\mathcal{E}_\pi)$;
- (4) the sequence $\{\mu_1^n : n \in \mathbb{N}\}$ converges to μ_1 in the sense of well-known weak star topology over $\mathcal{P}(Y_1)$, i.e. the sequence $\{\int_{Y_1} f d\mu_1^n : n \in \mathbb{N}\}$ converges to $\int_{Y_1} f d\mu_1$ for any $f \in C(Y_1)$.

In fact, the equivalence of (1) and (2) follow from the ideas of [44, Lemma 2.1], the equivalence of (2) and (3) is obvious (just note that \mathcal{E}_π is a non-empty compact subset of the compact metric space $Y_2 \times Y_1$), the equivalence of (3) and (4) is natural (just note that \mathcal{E}_π is identical to Y_1 by the natural homeomorphism $(y_2, y_1) \mapsto y_1$). From the above arguments, as topological spaces, $\mathcal{P}_{\nu_2}(\mathcal{E}_\pi)$ is identical to $\mathcal{P}_{\nu_2}(Y_1)$ by the natural homeomorphism which is also a homeomorphism from $\mathcal{P}_{\nu_2}(\mathcal{E}_\pi, G)$ onto $\mathcal{P}_{\nu_2}(Y_1, G)$. Moreover, it is not hard to check the following observations (note that each $\nu_1 \in \mathcal{P}(Y_1, G)$ may be viewed as an element from $\mathcal{P}_{\pi\nu_1}(\mathcal{E}_\pi, G)$):

- (1) If $\mathcal{V} \in \mathbf{C}_{Y_1}^o$ then by Theorem 6.9 we see that $\mathcal{V}^\pi \in \mathbf{C}_{\mathcal{E}_\pi}^o$ is factor excellent (by the construction of \mathcal{V}^π , i.e. (13.1) and (13.2)).

(2) For each $\nu_1 \in \mathcal{P}(Y_1, G)$, $\nu_1(\mathbf{D}^\pi) = \nu_1(\mathbf{D})$ ($\nu_1(\mathbf{D})$ is defined similarly) and

$$h_{\nu_1}^{(r)}(\mathbf{F}^\pi, \mathcal{V}^\pi) (= h_{\nu_1, +}^{(r)}(\mathbf{F}^\pi, \mathcal{V}^\pi)) = h_{\nu_1}(G, \mathcal{V}|\pi),$$

hence, by Theorem 4.11 one has $h_{\nu_1}^{(r)}(\mathbf{F}^\pi) = h_{\nu_1}(G, Y_1|\pi)$.

(3) For each $\nu_2 \in \mathcal{P}(Y_2, G)$,

$$\nu_2 h_{\text{top}}^{(r)}(\mathbf{F}^\pi, \mathcal{V}^\pi) = \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{Y_2} \log N(\mathcal{V}_{F_n}, \pi^{-1}(y_2)) d\nu_2(y_2),$$

$$\nu_2 P_{\mathcal{E}_\pi}(\mathbf{D}^\pi, \mathcal{V}^\pi, \mathbf{F}^\pi) = \lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{Y_2} \log P_\pi(y_2, \mathbf{D}, F_n, \mathcal{V}) d\nu_2(y_2).$$

Thus, in the above notations from the definitions one sees directly that $E_n^{\nu_1}(Y_1, G|\pi) = E_{n, \nu_1}^{(r)}(\mathcal{E}_\pi, G)$ for each $\nu_1 \in \mathcal{P}(Y_1, G)$.

In particular, we have another version of Theorem 7.1 in the setting of given a factor map between TDSs which is stated as follows.

Theorem 13.1. *Let $\pi : (Y_1, G) \rightarrow (Y_2, G)$ be a factor map between TDSs and $\mathcal{V} \in \mathbf{C}_{Y_1}^\circ$, $\nu_2 \in \mathcal{P}(Y_2, G)$. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{|F_n|} \int_{Y_2} \log N(\mathcal{V}_{F_n}, \pi^{-1}(y_2)) d\nu_2(y_2) = \max_{\nu_1 \in \mathcal{P}_{\nu_2}(Y_1, G)} h_{\nu_1}(G, \mathcal{V}|\pi).$$

Remark 13.2. *This result may be viewed as a local version of the Inner Variational Principle [23, Theorem 4] (see also [48, Theorem 2.1]) in the general case of our setting. For the case of \mathbb{Z} -actions see for example [73, Theorem 4.2.15].*

Let X_1, X_2 be topological spaces. Recall that the map $\pi : X_1 \rightarrow X_2$ is *open* if $\pi(U)$ is an open subset of X_2 whenever U is an open subset of X_1 .

From the definitions, it is not hard to obtain:

Proposition 13.3. *Let $\pi : (Y_1, G) \rightarrow (Y_2, G)$ be a factor map between TDSs, $\nu_2 \in \mathcal{P}(Y_2, G)$ and $n \in \mathbb{N} \setminus \{1\}$. Then*

(13.3)

$$\nu_2 E_n^{(r)}(\mathcal{E}_\pi) \subseteq \{(x_1, \dots, x_n) \in Y_1^n \setminus \Delta_n(Y_1) : \pi(x_1) = \dots = \pi(x_n) \in \text{supp}(\nu_2)\}.$$

If, additionally, π is open, then the identity holds.

Proof. We first establish (13.3). Let $(x_1, \dots, x_n) \in \nu_2 E_n^{(r)}(\mathcal{E}_\pi)$. By the definition, for each $m \in \mathbb{N}$ there exist $y_2^m \in Y_2$ and $(x_1^m, \dots, x_n^m) \in Y_1^n$ such that $(y_2^m, x_i^m) \in \mathcal{E}_\pi$ and the distance between x_i^m and x_i is at most $\frac{1}{m}$ for each $i = 1, \dots, n$. Without loss of generality (by selecting a sub-sequence if necessary) we may assume that the sequence $\{y_2^m : m \in \mathbb{N}\}$ converges to $y_2 \in Y$, and so it is easy to check $\pi(x_1) = \dots = \pi(x_n) = y_2$. Now we aim to prove (13.3) by proving $y_2 \in \text{supp}(\nu_2)$. Assume the contrary that $y_2 \notin \text{supp}(\nu_2)$. Obviously, once $m \in \mathbb{N}$ is large enough, if V_i is a closed neighborhood of x_i with diameter at most $\frac{1}{m}$ for each $i = 1, \dots, n$ such that $\mathcal{V} = \{V_1^c, \dots, V_n^c\} \in \mathbf{C}_{Y_1}^\circ$, then $\bigcup_{i=1}^n V_i \subseteq \pi^{-1}(Y_2 \setminus \text{supp}(\nu_2))$ and so

$$\{y \in Y_2 : \prod_{i=1}^n \{y\} \times V_i \cap \mathcal{E}_\pi^n \neq \emptyset\} \subseteq \bigcap_{i=1}^n \pi(V_i) \subseteq Y_2 \setminus \text{supp}(\nu_2),$$

a contradiction to $(x_1, \dots, x_n) \in \nu_2 E_n^{(r)}(\mathcal{E}_\pi)$, as $\nu_2(Y_2 \setminus \text{supp}(\nu_2)) = 0$.

Now we assume that π is open. Let $(x_1, \dots, x_n) \in Y_1^n \setminus \Delta_n(Y_1)$ with $\pi(x_1) = \dots = \pi(x_n) \in \text{supp}(\nu_2)$. Observe that once V_i is a closed neighborhood of x_i

for each $i = 1, \dots, n$ then $\bigcap_{i=1}^n \pi(V_i)$ is a closed neighborhood of $\pi(x_1)$ (using the openness of π), which implies $\nu_2(\bigcap_{i=1}^n \pi(V_i)) > 0$ (as $\pi(x_1) \in \text{supp}(\nu_2)$) and hence $(x_1, \dots, x_n) \in_{\nu_2} E_n^{(r)}(\mathcal{E}_\pi)$. This finishes the proof. \square

Before proceeding, we need the following result.

Theorem 13.4. *Let $\pi : (Y_1, G) \rightarrow (Y_2, G)$ be a factor map between TDSs, $\mathcal{V} \in \mathbf{C}_{Y_1}^o$ and $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\}$ a monotone sub-additive G -invariant family in $C(Y_1)$. Assume that \mathbf{D} satisfies:*

(\heartsuit) for any given sequence $\{\nu_n : n \in \mathbb{N}\} \subseteq \mathcal{P}(Y_1)$, set $\mu_n = \frac{1}{|F_n|} \sum_{g \in F_n} g\nu_n$ for each $n \in \mathbb{N}$, then there always exists some sub-sequence $\{n_j : j \in \mathbb{N}\} \subseteq \mathbb{N}$ such that the sequence $\{\mu_{n_j} : j \in \mathbb{N}\}$ converges to some $\mu \in \mathcal{P}(Y_1)$ (and so $\mu \in \mathcal{P}(Y_1, G)$) and

$$\limsup_{j \rightarrow \infty} \frac{1}{|F_{n_j}|} \int_{Y_1} d_{F_{n_j}}(y_1) d\nu_{n_j}(y_1) \leq \mu(\mathbf{D}).$$

Then

$$P_\pi(\mathbf{D}, \mathcal{V}) = \max_{\nu_2 \in \mathcal{P}(Y_2, G)} \nu_2 P_{\mathcal{E}_\pi}(\mathbf{D}^\pi, \mathcal{V}^\pi, \mathbf{F}^\pi) = \max_{\nu_1 \in \mathcal{P}(Y_1, G)} [h_{\nu_1}(G, \mathcal{V}|\pi) + \nu_1(\mathbf{D})].$$

In particular,

$$h_{top}(G, \mathcal{V}|\pi) = \max_{\nu_2 \in \mathcal{P}(Y_2, G)} \nu_2 h_{top}^{(r)}(\mathbf{F}^\pi, \mathcal{V}^\pi) = \max_{\nu_1 \in \mathcal{P}(Y_1, G)} h_{\nu_1}(G, \mathcal{V}|\pi).$$

Moreover, one has

$$P_\pi(\mathbf{D}) = \sup_{\nu_1 \in \mathcal{P}(Y_1, G)} [h_{\nu_1}(G, Y_1|\pi) + \nu_1(\mathbf{D})]$$

and so

$$h_{top}(G, Y_1|\pi) = \sup_{\nu_1 \in \mathcal{P}(Y_1, G)} h_{\nu_1}(G, Y_1|\pi).$$

Proof. The proof follows the ideas from §8 (see also for example [35, 37, 55, 73, 74] and the references in them). As the process is similar, we shall present the outline of the proof and skip some details (we should note that many results in §7 can be obtained in the setting of this section with a slight modification in the proves of them).

Observe that $\mathcal{V}^\pi \in \mathbf{C}_{\mathcal{E}_\pi}^o$ is factor excellent, and it is not hard to check that \mathbf{D}^π satisfies the assumption of (\spadesuit) and $(Y_2, \mathcal{B}_{Y_2}, \nu_2)$ is a Lebesgue space for each $\nu_2 \in \mathcal{P}(Y_2, G)$ (as \mathbf{D} satisfies the assumption of (\heartsuit)). In particular, Theorem 7.1 holds for $\mathbf{F}^\pi, \mathcal{V}^\pi, \mathbf{D}^\pi$ and $(Y_2, \mathcal{B}_{Y_2}, \nu_2)$ for each $\nu_2 \in \mathcal{P}(Y_2, G)$.

Thus, to complete our proof, we only need to find $\nu_1 \in \mathcal{P}(Y_1, G)$ with

$$(13.4) \quad h_{\nu_1}(G, \mathcal{V}|\pi) + \nu_1(\mathbf{D}) \geq P_\pi(\mathbf{D}, \mathcal{V}).$$

First, we assume that the space Y_1 is zero-dimensional. By Lemma 6.1 the family $\mathbf{P}_c(\mathcal{V})$ is countable and we let $\{\alpha_l : l \in \mathbb{N}\}$ denote an enumeration of this family. Then each $\alpha_l, l \in \mathbb{N}$ is finer than \mathcal{V} and, for each $\nu_1 \in \mathcal{P}(Y_1, G)$,

$$(13.5) \quad h_{\nu_1}(G, \mathcal{V}|\pi) = \inf_{l \in \mathbb{N}} h_{\nu_1}(G, \alpha_l|\pi).$$

Observe that by our assumptions $|F_n| \geq n$ for each $n \in \mathbb{N}$ and let $n \in \mathbb{N}$ be fixed. By similar reasoning to Lemma 8.3 (in fact, the reasoning of this case is much simpler than it in Lemma 8.3) one sees that there exist $x_n \in Y_2$ and a non-empty finite subset $B_n \subseteq \pi^{-1}(x_n)$ such that

$$(13.6) \quad \sum_{y \in B_n} e^{d_{F_n}(y)} \geq \frac{1}{n} \left[\sup_{y_2 \in Y_2} P_\pi(y_2, \mathbf{D}, F_n, \mathcal{V}) - M \right]$$

with

$$M = \frac{1}{2} e^{-\max_{y_1 \in Y_1} |d_{F_n}(y_1)|}$$

and each atom of $(\alpha_l)_{F_n}$ contains at most one point of B_n for each $l = 1, \dots, n$. Now let

$$(13.7) \quad \nu_n = \sum_{y \in B_n} \frac{e^{d_{F_n}(y)} \delta_y}{\sum_{x \in B_n} e^{d_{F_n}(x)}} \in \mathcal{P}(Y_1) \text{ and } \mu_n = \frac{1}{|F_n|} \sum_{g \in F_n} g \nu_n \in \mathcal{P}(Y_1).$$

By (\heartsuit) , we can choose a sub-sequence $\{n_j : j \in \mathbb{N}\} \subseteq \mathbb{N}$ such that the sequence $\{\mu_{n_j} : j \in \mathbb{N}\}$ converges to $\mu \in \mathcal{P}(Y_1, G)$ and

$$(13.8) \quad \limsup_{j \rightarrow \infty} \frac{1}{|F_{n_j}|} \int_{Y_1} d_{F_{n_j}}(y_1) d\nu_{n_j}(y_1) \leq \mu(\mathbf{D}).$$

Now fix any $l \in \mathbb{N}$ and let $n > l$. By the construction of B_n, ν_n one has

$$(13.9) \quad H_{\nu_n}((\alpha_l)_{F_n}|\pi) = H_{\nu_n}((\alpha_l)_{F_n}) = - \sum_{y \in B_n} \frac{e^{d_{F_n}(y)}}{\sum_{x \in B_n} e^{d_{F_n}(x)}} \log \frac{e^{d_{F_n}(y)}}{\sum_{x \in B_n} e^{d_{F_n}(x)}},$$

and so

$$\begin{aligned} & \log \sup_{y_2 \in Y_2} P_\pi(y_2, \mathbf{D}, F_n, \mathcal{V}) - \log(2n) \\ & \leq \log \left[\sup_{y_2 \in Y_2} P_\pi(y_2, \mathbf{D}, F_n, \mathcal{V}) - M \right] - \log n \\ & \leq \log \sum_{y \in B_n} e^{d_{F_n}(y)} \text{ (using (13.6))} \\ & = H_{\nu_n}((\alpha_l)_{F_n}|\pi) + \sum_{y \in B_n} \frac{e^{d_{F_n}(y)} d_{F_n}(y)}{\sum_{x \in B_n} e^{d_{F_n}(x)}} \text{ (using (13.9))} \\ (13.10) \quad & = H_{\nu_n}((\alpha_l)_{F_n}|\pi) + \int_{Y_1} d_{F_n}(y_1) d\nu_n(y_1) \text{ (using (13.7)).} \end{aligned}$$

Observe that using Lemma 8.4 and Lemma 8.5 one has

$$\begin{aligned} H_{\nu_n}((\alpha_l)_{F_n}|\pi) & \leq \sum_{g \in F_n} \frac{1}{|B|} H_{\nu_n}((\alpha_l)_{Bg}|\pi) + |F_n \setminus \{g \in G : B^{-1}g \subseteq F_n\}| \cdot \log |\alpha_l| \\ & = \sum_{g \in F_n} \frac{1}{|B|} H_{g\nu_n}((\alpha_l)_B|\pi) + |F_n \setminus \{g \in G : B^{-1}g \subseteq F_n\}| \cdot \log |\alpha_l| \\ (13.11) \quad & \leq |F_n| \frac{1}{|B|} H_{\mu_n}((\alpha_l)_B|\pi) + |F_n \setminus \{g \in G : B^{-1}g \subseteq F_n\}| \cdot \log |\alpha_l| \end{aligned}$$

for each $B \in \mathcal{F}_G$. Combining (13.11) with (13.8) and (13.10) we obtain (observe that the partition α_l is clopen)

$$(13.12) \quad P_\pi(\mathbf{D}, \mathcal{V}) \leq \frac{1}{|B|} H_\mu((\alpha_l)_B | \pi) + \mu(\mathbf{D}).$$

Now taking the infimum over $B \in \mathcal{F}_G$ we get (using (3.3))

$$P_\pi(\mathbf{D}, \mathcal{V}) \leq h_\mu(G, \alpha_l | \pi) + \mu(\mathbf{D}).$$

Finally, letting l range over \mathbb{N} one has $P_\pi(\mathbf{D}, \mathcal{V}) \leq h_\mu(G, \mathcal{V} | \pi) + \mu(\mathbf{D})$ (using (13.5)).

Now we consider the general case. Note that there always exists a factor map $\phi : (X, G) \rightarrow (Y_1, G)$ between TDSs, where X is a zero-dimensional space (see for example the proof of Proposition 6.7 or [37, Theorem 5.1]). Then by the above discussions there exists $\nu \in \mathcal{P}(X, G)$ such that

$$h_\nu(G, \phi^{-1} \mathcal{V} | \pi \circ \phi) + \nu(\mathbf{D} \circ \phi) \geq P_{\pi \circ \phi}(\mathbf{D} \circ \phi, \phi^{-1} \mathcal{V}),$$

where the family $\mathbf{D} \circ \phi$ is defined naturally. Set $\eta = \phi\nu$. It is not hard to check that $\eta \in \mathcal{P}(Y_1, G)$ and $h_\eta(G, \mathcal{V} | \pi) + \eta(\mathbf{D}) \geq P_\pi(\mathbf{D}, \mathcal{V})$. This claims (13.4) in the general case, which ends our proof. \square

We should remark that:

- (1) Similar to Remark 7.3 (see also Remark 7.6), we can apply this discussion to each $f \in C(Y_1)$.
- (2) Discussion and conclusions similar to that in §9 hold for the assumption (\heartsuit) .
- (3) As in our discussions in §10, if G admits a tiling Følner sequence, then we can discuss Theorem 13.1 and Theorem 13.4 for any sub-additive G -invariant family $\mathbf{D} \subseteq C(Y_1)$. In particular, [74, Theorem 4.5] may be viewed as a special case of our result. In fact, variations of Theorem 13.1 and Theorem 13.4 are stronger than results obtained in [35, 37, 73, 74] (i.e. local variational principles for entropy) even in the special case of \mathbb{Z} -actions or topological dynamical G -systems.

With the help of Theorem 12.3, Theorem 12.4, Lemma 12.13, Theorem 12.16 and Proposition 13.3, as an application of Theorem 13.4 we can prove (the proof follows ideas from [28, 32, 34, 36, 37] and is quite standard, and so we shall omit it, for details see [28, 32, 34, 36, 37] or §12 of the paper):

Theorem 13.5. *Let $\pi : (Y_1, G) \rightarrow (Y_2, G)$ be a factor map between TDSs and $\nu \in \mathcal{P}(Y_1, G)$, $\nu_2 \in \mathcal{P}(Y_2, G)$, $n \in \mathbb{N} \setminus \{1\}$. Then*

$$\begin{aligned} E_{n, \nu}^{(r)}(\mathcal{E}_\pi, G) &= E_n^\nu(Y_1, G | \pi) \\ &\subseteq \{(x_1, \dots, x_n) \in \text{supp}(\nu)^n \setminus \Delta_n(Y_1) : \pi(x_1) = \dots = \pi(x_n)\}, \\ \nu_2 E_n^{(r)}(\mathcal{E}_\pi, G) &= \bigcup_{\nu_1 \in \mathcal{P}_{\nu_2}(Y_1, G)} E_n^{\nu_1}(Y_1, G | \pi) \\ &\subseteq \{(x_1, \dots, x_n) \in \left(\bigcup_{\nu_1 \in \mathcal{P}_{\nu_2}(Y_1, G)} \text{supp}(\nu_1) \right)^n \setminus \Delta_n(Y_1) : \pi(x_1) = \dots = \pi(x_n)\}, \\ E_n(Y_1, G | \pi) &= \bigcup_{\eta \in \mathcal{P}(Y_2, G)} \eta E_n^{(r)}(\mathcal{E}_\pi, G) = \bigcup_{\mu \in \mathcal{P}(Y_1, G)} E_n^\mu(Y_1, G | \pi). \end{aligned}$$

In particular, there exists $\mu \in \mathcal{P}(Y_1, G)$ such that

$$E_n(Y_1, G|\pi) = {}_{\pi\mu} E_n^{(r)}(\mathcal{E}_\pi, G) = E_n^\mu(Y_1, G|\pi).$$

Let (Y, G) be a TDS. Denote by $\text{supp}(Y, G)$, the *support* of (Y, G) , the set of $\bigcup_{\mu \in \mathcal{P}(Y, G)} \text{supp}(\mu)$. Observe that $\text{supp}(Y, G) = \text{supp}(\nu)$ for some $\nu \in \mathcal{P}(Y, G)$.

Combining with Proposition 12.6, Proposition 12.7 and Theorem 12.15, by the natural correspondence introduced in the beginning of this section we obtain:

Proposition 13.6. *Let $\pi : (Y_1, G) \rightarrow (Y_2, G)$ be a factor map between TDSs and $\mu \in \mathcal{P}(Y_1, G)$, $n \in \mathbb{N} \setminus \{1\}$. Then both $E_n(Y_1, G|\pi)$ and $E_n^\mu(Y_1, G|\pi)$ are G -invariant subsets of Y_1^n , in fact, $E_n(Y_1, G|\pi) \neq \emptyset$ if and only if $h_{\text{top}}(G, Y_1|\pi) > 0$ and $E_n^\mu(Y_1, G|\pi) \neq \emptyset$ if and only if $h_\mu(G, Y_1|\pi) > 0$, moreover,*

$$E_n(Y_1, G|\pi) \subseteq \{(x_1, \dots, x_n) \in \text{supp}(Y_1, G)^n : \pi(x_1) = \dots = \pi(x_n)\},$$

$$E_n^\mu(Y_1, G|\pi) = \text{supp}(\lambda_n^{\pi^{-1}\mathcal{B}_{Y_2}}(\mu)) \setminus \Delta_n(Y_1).$$

Moreover, using Proposition 12.5 one has:

Proposition 13.7. *Let $\pi_1 : (Y_1, G) \rightarrow (Y_2, G)$ and $\pi_2 : (Y_2, G) \rightarrow (Y_3, G)$ be factor maps between TDSs and $\nu_1 \in \mathcal{P}(Y_1, G)$, $\nu_2 = \pi_1\nu_1 \in \mathcal{P}(Y_2, G)$, $n \in \mathbb{N} \setminus \{1\}$. Then*

- (1) $E_n^{\nu_2}(Y_2, G|\pi_2) \subseteq (\pi_1 \times \dots \times \pi_1)E_n^{\nu_1}(Y_1, G|\pi_2 \circ \pi_1) \subseteq E_n^{\nu_2}(Y_2, G|\pi_2) \cup \Delta_n(Y_2)$.
- (2) $E_n(Y_2, G|\pi_2) \subseteq (\pi_1 \times \dots \times \pi_1)E_n(Y_1, G|\pi_2 \circ \pi_1) \subseteq E_n(Y_2, G|\pi_2) \cup \Delta_n(Y_2)$.
- (3) $E_n^{\nu_1}(Y_1, G|\pi_2 \circ \pi_1) \subseteq E_n^{\nu_1}(Y_1, G|\pi_1)$ and $E_n(Y_1, G|\pi_2 \circ \pi_1) \subseteq E_n(Y_1, G|\pi_1)$.

As the notions of entropy tuples in both settings cover the standard definitions for \mathbb{Z} -actions and more generally for an infinite countable discrete amenable group action. Thus, our Theorem 13.5, Proposition 13.6 and Proposition 13.7 include many recent results in local entropy theory (see [4, 6, 28, 32, 34, 36, 37] and the references in them for the details of those results).

REFERENCES

1. L. M. Abramov and V. A. Rohlin, *Entropy of a skew product of mappings with invariant measure*, Vestnik Leningrad. Univ. **17** (1962), no. 7, 5–13. MR 0140660 (25 #4076)
2. L. Arnold, *Random dynamical systems*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998. MR 1723992 (2000m:37087)
3. F. Blanchard, *Fully positive topological entropy and topological mixing*, Symbolic dynamics and its applications (New Haven, CT, 1991), Contemp. Math., vol. 135, Amer. Math. Soc., Providence, RI, 1992, pp. 95–105. MR 1185082 (93k:58134)
4. ———, *A disjointness theorem involving topological entropy*, Bull. Soc. Math. France **121** (1993), no. 4, 465–478. MR 1254749 (95e:54050)
5. F. Blanchard, E. Glasner, and B. Host, *A variation on the variational principle and applications to entropy pairs*, Ergodic Theory Dynam. Systems **17** (1997), no. 1, 29–43. MR 1440766 (98k:54073)
6. F. Blanchard, B. Host, A. Maass, S. Martinez, and D. J. Rudolph, *Entropy pairs for a measure*, Ergodic Theory Dynam. Systems **15** (1995), no. 4, 621–632. MR 1346392 (96m:28024)
7. F. Blanchard and Y. Lacroix, *Zero entropy factors of topological flows*, Proc. Amer. Math. Soc. **119** (1993), no. 3, 985–992. MR 1155593 (93m:54066)
8. T. Bogenschütz, *Entropy, pressure, and a variational principle for random dynamical systems*, Random Comput. Dynam. **1** (1992/93), no. 1, 99–116. MR 1181382 (93k:28023)
9. ———, *Equilibrium states for random dynamical systems*, Ph.D. Thesis, Universitat Bremen, 1993.
10. T. Bogenschütz and V. M. Gundlach, *Ruelle’s transfer operator for random subshifts of finite type*, Ergodic Theory Dynam. Systems **15** (1995), no. 3, 413–447. MR 1336700 (96m:58133)
11. Y. L. Cao, D. J. Feng, and W. Huang, *The thermodynamic formalism for sub-additive potentials*, Discrete Contin. Dyn. Syst. **20** (2008), no. 3, 639–657. MR 2373208 (2008k:37072)
12. C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Lecture Notes in Mathematics, Vol. 580, Springer-Verlag, Berlin, 1977. MR 0467310 (57 #7169)
13. A. Connes, J. Feldman, and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergodic Theory Dynamical Systems **1** (1981), no. 4, 431–450 (1982). MR 662736 (84h:46090)
14. H. Crauel, *Random probability measures on Polish spaces*, Stochastics Monographs, vol. 11, Taylor & Francis, London, 2002. MR 1993844 (2004e:60005)
15. H. Crauel, A. Debussche, and F. Flandoli, *Random attractors*, J. Dynam. Differential Equations **9** (1997), no. 2, 307–341. MR 1451294 (98c:60066)
16. Alexandre I. Danilenko, *Entropy theory from the orbital point of view*, Monatsh. Math. **134** (2001), no. 2, 121–141. MR 1878075 (2002j:37011)
17. Alexandre I. Danilenko and K. K. Park, *Generators and Bernoullian factors for amenable actions and cocycles on their orbits*, Ergodic Theory Dynam. Systems **22** (2002), no. 6, 1715–1745. MR 1944401 (2004f:37006)
18. A. H. Dooley and V. Ya. Golodets, *The spectrum of completely positive entropy actions of countable amenable groups*, J. Funct. Anal. **196** (2002), no. 1, 1–18. MR 1941988 (2003m:37006)
19. A. H. Dooley, V. Ya. Golodets, and G. H. Zhang, *Sub-additive ergodic theorems for countable amenable groups*, preprint, 2011.
20. A. H. Dooley and G. H. Zhang, *Co-induction in dynamical systems*, Ergodic Theory Dynamical Systems, to appear.
21. D. Dou, X. Ye, and G. H. Zhang, *Entropy sequences and maximal entropy sets*, Nonlinearity **19** (2006), no. 1, 53–74. MR 2191619 (2006i:37037)
22. T. Downarowicz, *Entropy in dynamical systems*, New Mathematical Monographs, vol. 18, Cambridge University Press, Cambridge, 2011.
23. T. Downarowicz and J. Serafin, *Fiber entropy and conditional variational principles in compact non-metrizable spaces*, Fund. Math. **172** (2002), no. 3, 217–247. MR 1898686 (2003b:37027)
24. R. M. Dudley, *Real analysis and probability*, Cambridge Studies in Advanced Mathematics, vol. 74, Cambridge University Press, Cambridge, 2002, Revised reprint of the 1989 original. MR 1932358 (2003h:60001)

25. William R. Emerson, *The pointwise ergodic theorem for amenable groups*, Amer. J. Math. **96** (1974), 472–487. MR 0354926 (50 #7403)
26. I. V. Evstigneev, *Measurable selection theorems and probabilistic models of control in general topological spaces*, Mat. Sb. (N.S.) **131(173)** (1986), no. 1, 27–39, 126. MR 868599 (88b:28021)
27. H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, N.J., 1981, M. B. Porter Lectures. MR 603625 (82j:28010)
28. E. Glasner, *A simple characterization of the set of μ -entropy pairs and applications*, Israel J. Math. **102** (1997), 13–27. MR 1489099 (98k:54076)
29. ———, *Ergodic theory via joinings*, Mathematical Surveys and Monographs, vol. 101, American Mathematical Society, Providence, RI, 2003. MR 1958753 (2004c:37011)
30. E. Glasner, J.-P. Thouvenot, and B. Weiss, *Entropy theory without a past*, Ergodic Theory Dynam. Systems **20** (2000), no. 5, 1355–1370. MR 1786718 (2001h:37011)
31. E. Glasner and B. Weiss, *On the interplay between measurable and topological dynamics*, Handbook of dynamical systems. Vol. 1B, Elsevier B. V., Amsterdam, 2006, pp. 597–648. MR 2186250 (2006i:37005)
32. E. Glasner and X. Ye, *Local entropy theory*, Ergodic Theory Dynam. Systems **29** (2009), no. 2, 321–356. MR 2486773 (2010k:37023)
33. W. Huang, A. Maass, P. P. Romagnoli, and X. Ye, *Entropy pairs and a local Abramov formula for a measure theoretical entropy of open covers*, Ergodic Theory Dynam. Systems **24** (2004), no. 4, 1127–1153. MR 2085906 (2005e:37027)
34. W. Huang and X. Ye, *A local variational relation and applications*, Israel J. Math. **151** (2006), 237–279. MR 2214126 (2006k:37033)
35. W. Huang, X. Ye, and G. H. Zhang, *A local variational principle for conditional entropy*, Ergodic Theory Dynam. Systems **26** (2006), no. 1, 219–245. MR 2201946 (2006j:37015)
36. ———, *Relative entropy tuples, relative U.P.E. and C.P.E. extensions*, Israel J. Math. **158** (2007), 249–283. MR 2342467 (2008h:37016)
37. ———, *Local entropy theory for a countable discrete amenable group action*, J. Funct. Anal. **261** (2011), no. 4, 1028–1082.
38. W. Huang and Y. Yi, *A local variational principle of pressure and its applications to equilibrium states*, Israel J. Math. **161** (2007), 29–74. MR 2350155 (2008i:37013)
39. S. Kakutani, *Random ergodic theorems and Markoff processes with a stable distribution*, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950 (Berkeley and Los Angeles), University of California Press, 1951, pp. 247–261. MR 0044773 (13,476a)
40. D. Kerr and H. F. Li, *Independence in topological and C^* -dynamics*, Math. Ann. **338** (2007), no. 4, 869–926. MR 2317754 (2009a:46126)
41. K. Khanin and Y. Kifer, *Thermodynamic formalism for random transformations and statistical mechanics*, Sinai’s Moscow Seminar on Dynamical Systems, Amer. Math. Soc. Transl. Ser. 2, vol. 171, Amer. Math. Soc., Providence, RI, 1996, pp. 107–140. MR 1359097 (96j:58136)
42. J. C. Kieffer, *A generalized Shannon-McMillan theorem for the action of an amenable group on a probability space*, Ann. Probability **3** (1975), no. 6, 1031–1037. MR 0393422 (52 #14232)
43. Y. Kifer, *Ergodic theory of random transformations*, Progress in Probability and Statistics, vol. 10, Birkhäuser Boston Inc., Boston, MA, 1986. MR 884892 (89c:58069)
44. ———, *On the topological pressure for random bundle transformations*, Topology, ergodic theory, real algebraic geometry, Amer. Math. Soc. Transl. Ser. 2, vol. 202, Amer. Math. Soc., Providence, RI, 2001, pp. 197–214. MR 1819189 (2002c:37047)
45. Y. Kifer and P. D. Liu, *Random dynamics*, Handbook of dynamical systems. Vol. 1B, Elsevier B. V., Amsterdam, 2006, pp. 379–499. MR 2186245 (2008a:37002)
46. Y. Kifer and B. Weiss, *Generating partitions for random transformations*, Ergodic Theory Dynam. Systems **22** (2002), no. 6, 1813–1830. MR 1944406 (2003k:37007)
47. F. Ledrappier, *A variational principle for the topological conditional entropy*, Ergodic theory (Proc. Conf., Math. Forschungsinst., Oberwolfach, 1978), Lecture Notes in Math., vol. 729, Springer, Berlin, 1979, pp. 78–88. MR 550412 (80j:54011)
48. F. Ledrappier and P. Walters, *A relativised variational principle for continuous transformations*, J. London Math. Soc. (2) **16** (1977), no. 3, 568–576. MR 0476995 (57 #16540)
49. Z. Lian and K. N. Lu, *Lyapunov exponents and invariant manifolds for random dynamical systems in a Banach space*, Mem. Amer. Math. Soc. **206** (2010), no. 967, vi+106. MR 2674952

50. E. Lindenstrauss, *Pointwise theorems for amenable groups*, Invent. Math. **146** (2001), no. 2, 259–295. MR 1865397 (2002h:37005)
51. E. Lindenstrauss and B. Weiss, *Mean topological dimension*, Israel J. Math. **115** (2000), 1–24. MR 1749670 (2000m:37018)
52. P. D. Liu, *Dynamics of random transformations: smooth ergodic theory*, Ergodic Theory Dynam. Systems **21** (2001), no. 5, 1279–1319. MR 1855833 (2002g:37024)
53. ———, *A note on the entropy of factors of random dynamical systems*, Ergodic Theory Dynam. Systems **25** (2005), no. 2, 593–603. MR 2129111 (2008b:37004)
54. P. D. Liu and M. Qian, *Smooth ergodic theory of random dynamical systems*, Lecture Notes in Mathematics, vol. 1606, Springer-Verlag, Berlin, 1995. MR 1369243 (96m:58139)
55. Michał Misiurewicz, *A short proof of the variational principle for a \mathbf{Z}_+^N action on a compact space*, International Conference on Dynamical Systems in Mathematical Physics (Rennes, 1975), Soc. Math. France, Paris, 1976, pp. 147–157. Astérisque, No. 40. MR 0444904 (56 #3250)
56. J. Moulin Ollagnier, *Ergodic theory and statistical mechanics*, Lecture Notes in Mathematics, vol. 1115, Springer-Verlag, Berlin, 1985. MR 781932 (86h:28013)
57. J. Moulin Ollagnier and D. Pinchon, *Groupes pavables et principe variationnel*, Z. Wahrsch. Verw. Gebiete **48** (1979), no. 1, 71–79. MR 533007 (80g:28019)
58. ———, *The variational principle*, Studia Math. **72** (1982), no. 2, 151–159. MR 665415 (83j:28019)
59. D. S. Ornstein and B. Weiss, *Entropy and isomorphism theorems for actions of amenable groups*, J. Analyse Math. **48** (1987), 1–141. MR 910005 (88j:28014)
60. ———, *Subsequence ergodic theorems for amenable groups*, Israel J. Math. **79** (1992), no. 1, 113–127. MR 1195256 (94g:28024)
61. V. A. Rohlin, *On the fundamental ideas of measure theory*, Mat. Sbornik N.S. **25(67)** (1949), 107–150. MR 0030584 (11,18f)
62. V. A. Rohlin and Ja. G. Sinaĭ, *The structure and properties of invariant measurable partitions*, Dokl. Akad. Nauk SSSR **141** (1961), 1038–1041. MR 0152629 (27 #2604)
63. P. P. Romagnoli, *A local variational principle for the topological entropy*, Ergodic Theory Dynam. Systems **23** (2003), no. 5, 1601–1610. MR 2018614 (2004i:37030)
64. D. J. Rudolph and B. Weiss, *Entropy and mixing for amenable group actions*, Ann. of Math. (2) **151** (2000), no. 3, 1119–1150. MR 1779565 (2001g:37001)
65. A. M. Stepin and A. T. Tagi-Zade, *Variational characterization of topological pressure of the amenable groups of transformations*, Dokl. Akad. Nauk SSSR **254** (1980), no. 3, 545–549. MR 590147 (82a:28016)
66. S. M. Ulam and J. von Neumann, *Random ergodic theorems*, Bull. Amer. Math. Soc. **51** (1945), 660.
67. P. Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York, 1982. MR 648108 (84e:28017)
68. T. Ward and Q. Zhang, *The Abramov-Rokhlin entropy addition formula for amenable group actions*, Monatsh. Math. **114** (1992), no. 3-4, 317–329. MR 1203977 (93m:28023)
69. B. Weiss, *Monotileable amenable groups*, Topology, ergodic theory, real algebraic geometry, Amer. Math. Soc. Transl. Ser. 2, vol. 202, Amer. Math. Soc., Providence, RI, 2001, pp. 257–262. MR 1819193 (2001m:22014)
70. ———, *Actions of amenable groups*, Topics in dynamics and ergodic theory, London Math. Soc. Lecture Note Ser., vol. 310, Cambridge Univ. Press, Cambridge, 2003, pp. 226–262. MR 2052281 (2005d:37008)
71. X. Ye and G. H. Zhang, *Entropy points and applications*, Trans. Amer. Math. Soc. **359** (2007), no. 12, 6167–6186 (electronic). MR 2336322 (2008m:37026)
72. G. H. Zhang, *Relative entropy, asymptotic pairs and chaos*, J. London Math. Soc. (2) **73** (2006), no. 1, 157–172. MR 2197376 (2006k:37035)
73. ———, *Relativization and localization of dynamical properties*, Ph.D. Thesis, University of Science and Technology of China, <http://homepage.fudan.edu.cn/~zhanggh/english.pdf>, 2007.
74. ———, *Variational principles of pressure*, Discrete Contin. Dyn. Syst. **24** (2009), no. 4, 1409–1435. MR 2505712 (2010h:37064)
75. Y. Zhao and Y. L. Cao, *On the topological pressure of random bundle transformations in sub-additive case*, J. Math. Anal. Appl. **342** (2008), no. 1, 715–725. MR 2440833 (2010d:37060)

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